

A GENERALIZATION OF VORONOI'S REDUCTION THEORY AND ITS APPLICATION

MATHIEU DUTOUR SIKIRIĆ, ACHILL SCHÜRMANN, AND FRANK VALLENTIN

ABSTRACT. We consider Voronoi's reduction theory of positive definite quadratic forms which is based on Delone subdivision. We extend it to forms and Delone subdivisions having a prescribed symmetry group. Even more general, the theory is developed for forms which are restricted to a linear subspace in the space of quadratic forms. We apply the new theory to complete the classification of totally real thin algebraic number fields which was recently initiated by Bayer-Fluckiger and Nebe. Moreover, we apply it to construct new best known sphere coverings in dimensions $9, \dots, 15$.

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1. INTRODUCTION

In this paper we generalize a classical reduction theory for positive definite quadratic forms due to Voronoi [Vor08]. His theory gives in particular an algorithm to classify Delone subdivisions of \mathbb{R}^d with vertex-set \mathbb{Z}^d up to the action of $\mathrm{GL}_d(\mathbb{Z})$. For precise definitions of used terms and a brief description of the classical theory we refer to Section 2.

We present our generalization in Section 3. We extend the classical theory in two different directions. On the one hand, we generalize the theory from vertex-set \mathbb{Z}^d to general periodic vertex-sets. On the other hand we give an equivariant theory dealing with positive definite quadratic forms with a prescribed automorphism group $G \leq \mathrm{GL}_d(\mathbb{Z})$. This equivariant theory is an analogue of the theory of G -perfect forms by A.M. Bergé, J. Martinet and F. Sigrist, [BMS92], which was motivated by the search of good packing lattices with prescribed symmetries. In fact, as in their case, our theory can be developed in the more general context of a linear subspace T of quadratic forms.

As in the classical theory, where T is the space of all quadratic forms, we get a polyhedral subdivision of the space of positive definite forms in T into generic T -secondary cones (in the classical case also called L -type domains), which contain those forms giving the same Delone subdivision. In contrast to the classical theory these generic subdivisions are no longer triangulations. In the equivariant theory, we have only finitely many T -secondary cones up to the action of $\mathrm{GL}_d(\mathbb{Z})$. Our proof of this fact in Section 4 uses only the action of $\mathrm{GL}_d(\mathbb{Z})$ on a polyhedral subdivision of the space of positive definite quadratic forms. So it applies to the theory of G -perfect forms and it gives a unified view on both theories (see Remark 4.4).

We describe the theory in a way which allows us to work with it computationally. In particular we made some effort to reduce redundancies in the description of secondary cones (see Theorem 3.1). Also the transition from a Delone subdivision of a T -secondary cone to the Delone subdivision of a contiguous T -secondary cone, called a T -flip, is given explicitly (see Theorem 3.8). For it we define repartitioning polytopes, in which a polyhedral subdivision has to be replaced by another. As a nice byproduct we obtain an explicit description of flips which occur in the theory of equivariant secondary polytopes of regular subdivisions (of a polytope), recently introduced by Reiner [Rei02] (see Remark 3.9).

We are not the first who consider generalizations of Voronoi's theory. Periodic tilings and reduction theory of positive definite quadratic forms appear naturally in algebraic geometry in the study of degenerations of abelian varieties and compactifications of Siegel modular varieties. For these reasons, Delone subdivisions and Voronoi's reduction theory were used and studied by Mumford and Namikawa about 25 years ago, and by many algebraic geometers since then. In Section 5 we review generalizations which came up in this context and compare them to our work.

We apply our extension of Voronoi's theory to two different problems. We use it to finish the classification of totally real thin number fields, which was recently started by Nebe and Bayer–Fluckiger [BN05] (Section 8). We use it to construct new best known sphere coverings (Section 9). Both applications involve finding best lattice configurations with respect to a given Delone subdivision. A brief description of the problem and of the convex optimization tools we used is given in Section 6. Since

we are dealing with non-linear optimization problems we can usually only approximate the lattices we want to find. By using convex optimization duality and rational approximations we can give mathematical rigorous error bounds for the quality of approximated lattices. Algorithmic issues in the classification of T -secondary cones are addressed in Section 7.

So far, using the new theory for the lattice case, we found new best known lattice sphere coverings in dimensions $d = 9, \dots, 15$. Using the classical theory (cf. [SV06]) and new methods to enumerate all vertices of symmetric Voronoi cells efficiently (cf. [DSV07]), we found new best known coverings in dimensions $d = 6, 7, 8$ and $d = 17, 19, 20, 21$ as well. A complete list of the best known values is given in Table 2 in Section 9. With the exception of dimension $d = 6$ and $d = 7$, we do not think that these lattice coverings are optimal.

Furthermore, we strongly believe that our extension of Voronoi's classical theory to periodic sets is a first step towards the construction of non-lattice coverings, which are less dense than any lattice covering.

2. BACKGROUND: LATTICES, PQFs AND DELONE SUBDIVISIONS

We start with basic definitions and basic results and some background on Voronoi's reduction theory. In the first section we introduce lattices and positive definite quadratic forms, PQFs from now on. For further reading we refer to [CS88] and [GL87]. In the second section we introduce Delone polyhedra and Delone subdivisions. For further reading about concepts related to polyhedra we refer to [Zie97]. In the third section we give a very brief review on Voronoi's results in [Vor08].

Let \mathbb{R}^d be the d -dimensional Euclidean space with column vectors $\mathbf{x} = (x_1, \dots, x_d)^t$ and norm $\|\mathbf{x}\| = \sqrt{\mathbf{x}^t \mathbf{x}}$.

2.1. Lattices and PQFs. A d -dimensional *lattice* L in \mathbb{R}^d is a discrete subgroup $L = \mathbb{Z}\mathbf{v}_1 + \dots + \mathbb{Z}\mathbf{v}_d$ with linearly independent $\mathbf{v}_i \in \mathbb{R}^d$. The family $(\mathbf{v}_1, \dots, \mathbf{v}_d)$ is called a *basis* of L . To it we associate the positive definite symmetric Gram matrix $Q_B = B^t B$, where $B \in \text{GL}_d(\mathbb{R})$ is the invertible matrix whose i -th column is \mathbf{v}_i , and $L = B\mathbb{Z}^d$.

Given on the other hand a positive definite symmetric matrix Q , there exists a matrix $B \in \text{GL}_d(\mathbb{R})$ with $Q = B^t B$. The matrix B is uniquely determined up to orthogonal transformations. Any other $A \in \text{GL}_d(\mathbb{R})$ with $A\mathbb{Z}^d = B\mathbb{Z}^d$ can be written as $A = BU$ with $U \in \text{GL}_d(\mathbb{Z})$. This relation yields $Q_A = A^t A = U^t Q_B U$. We say Q_A and Q_B are *arithmetical equivalent* in this case.

The space of real symmetric $d \times d$ matrices is denoted by \mathcal{S}^d . It is a $\binom{d+1}{2}$ -dimensional Euclidean space with inner product $\langle A, B \rangle = \text{trace}(AB)$. The subset $\mathcal{S}_{>0}^d$ of positive definite symmetric matrices is an open convex cone in \mathcal{S}^d . Abusing notation, we identify positive definite quadratic forms, PQFs from now on, and positive definite symmetric matrices by

$$Q[\mathbf{x}] = \mathbf{x}^t Q \mathbf{x} = \langle Q, \mathbf{x} \mathbf{x}^t \rangle.$$

The topological closure of $\mathcal{S}_{>0}^d$ in \mathcal{S}^d is the set $\mathcal{S}_{\geq 0}^d$ of all positive semidefinite matrices. By the relations above, $\mathcal{S}_{>0}^d$ can be identified with $\text{O}_d(\mathbb{R}) \backslash \text{GL}_d(\mathbb{R})$, where $\text{O}_d(\mathbb{R})$ denotes the subgroup of orthogonal $d \times d$ matrices in $\text{GL}_d(\mathbb{R})$. The group $\text{GL}_d(\mathbb{Z})$ acts on $\mathcal{S}_{>0}^d$ by $Q \mapsto U^t Q U$. Thus the set of *isometry classes* of d -dimensional lattices

(i.e. d -dimensional lattices up to orthogonal transformations) can be identified with $\mathcal{S}_{>0}^d / \mathrm{GL}_d(\mathbb{Z})$.

2.2. Delone polyhedra and Delone subdivisions. A *polyhedron* is a set in \mathbb{R}^d which can be represented as a finite intersection of closed half spaces, e.g. given by a system of linear inequalities. A *polytope* is the convex hull of finitely many points and by a theorem due to Minkowski and Weyl, polytopes are bounded polyhedra and vice versa. Given a discrete set $\Lambda \subset \mathbb{R}^d$, a polytope $P = \mathrm{conv}\{v_1, \dots, v_n\}$ with vertices $v_i \in \Lambda$ is called a *Delone polytope*, if there exists a center $c \in \mathbb{R}^d$ and a radius $r > 0$ such that the Euclidean distance between c to all points $v \in \Lambda$ satisfies $\|c - v\| \geq r$, with equality only for the vertices of P . The set of all Delone polytopes of a lattice forms a *polyhedral subdivision* of \mathbb{R}^d . This is a family of polyhedra, called *faces*, whose union is \mathbb{R}^d and which is closed with respect to intersections. Each face of the subdivision sharing relative interior points with another face of its dimension coincides with this face. Note, by a theorem of Gruber and Ryshkov [GR89], the latter *face-to-face* property holds whenever it holds for the facets (faces of co-dimension 1).

Our main interest is in vertex-sets Λ_L which are *periodic*, that is, a finite union of lattice translates of a lattice L , e.g., $\Lambda_L = \bigcup_{i=1}^m t_i + L$ with $t_i \in \mathbb{R}^d$ for $i = 1, \dots, m$. In many cases it is convenient to work with coordinates with respect to a given basis of L , say given by $B \in \mathrm{GL}_d(\mathbb{R})$. This means that we work with *standard periodic sets*

$$(1) \quad \Lambda = \bigcup_{i=1}^m t'_i + \mathbb{Z}^d$$

and norm defined by Q_B . Hence the norm of $x \in \mathbb{R}^d$ is given by $\sqrt{Q_B[x]}$. A polytope $P = \mathrm{conv}\{v_1, \dots, v_n\}$ with $v_i \in \Lambda$, is called a *Delone polytope* of Q_B and Λ , if there exists a $c \in \mathbb{R}^d$ and a real number r with $Q_B[c - v] \geq r^2$, where equality holds if and only if v is a vertex of P .

It is a bit more general (and in some situations convenient) to consider Delone subdivisions of positive semidefinite forms Q as proposed by Namikawa in [Nam76]. Positive semidefinite forms define seminorms on \mathbb{R}^d by $\sqrt{Q[x]}$. A (possibly unbounded) polyhedron $P = \mathrm{conv}\{v_1, v_2, \dots\}$ with $v_i \in \Lambda$ is then called a *Delone polyhedron* of Q , if there exists a $c \in \mathbb{R}^d$ and a real number r with $Q[c - v_i] = r^2$ for $i = 1, 2, \dots$ and $Q[c - v] > r^2$ for all $v \in \Lambda \setminus \{v_1, v_2, \dots\}$. Suppose for $Q \in \mathcal{S}_{\geq 0}^d$ there exists a matrix $U \in \mathrm{GL}_d(\mathbb{Z})$ and a $Q' \in \mathcal{S}_{>0}^{d'}$ with

$$(2) \quad U^t Q U = \begin{pmatrix} 0 & 0 \\ 0 & Q' \end{pmatrix} = \bar{Q}.$$

Then $d' = \mathrm{rank} Q$ and if P is a Delone polyhedron of Q then $U^{-1}P$ is a Delone polyhedron of \bar{Q} . Latter are of the form $\mathbb{R}^{d-d'} \times P'$ with $P' \in \mathbb{R}^{d'}$ a Delone polytope of Q' . The set of all forms being arithmetical equivalent to some positive semidefinite form \bar{Q} of the form (2), with Q' positive definite, is called the *rational closure* of $\mathcal{S}_{>0}^d$, denoted by $\tilde{\mathcal{S}}_{\geq 0}^d$. The importance of the rational closure is due to the following proposition.

Proposition 2.1. *For $Q \in \mathcal{S}_{\geq 0}^d$ exists a d -dimensional Delone polyhedron with respect to a standard periodic vertex-set if and only if $Q \in \tilde{\mathcal{S}}_{\geq 0}^d$.*

Although this proposition might be known we are not aware of a reference. In [Nam76, §2.1] Namikawa showed that every Q lying in the rational closure has a d -dimensional Delone polyhedron. For completeness we repeat his argument.

Proof. If $Q \in \tilde{\mathcal{S}}_{\geq 0}^d$, there exist d -dimensional Delone polyhedra which after a suitable transformation in $\mathrm{GL}_d(\mathbb{Z})$ are those of the form \bar{Q} in (2). If on the other hand $Q \notin \tilde{\mathcal{S}}_{\geq 0}^d$, then for all arithmetical equivalent forms \bar{Q} as in (2), the form $Q' \in \mathcal{S}_{\geq 0}^{d'}$ is not positive definite, hence $d' > \mathrm{rank}(Q)$. For an arithmetical equivalent form \bar{Q} with minimal d' there exists no rational, hence no integral vector in the kernel of Q' . As a consequence, for such a Q' we find for all $\mathbf{c} \in \mathbb{R}^{d'}$ and all $r > 0$ a $\mathbf{v} \in \mathbb{Z}^{d'}$ with $Q'[\mathbf{v} - \mathbf{c}] < r$ (see [Sie89, Lecture VI]). Therefore there do not exist d' -dimensional Delone polyhedra for Q' and $\mathbb{Z}^{d'}$, and hence no d -dimensional ones for Q and \mathbb{Z}^d as well. The same is true for Q and a standard periodic set Λ . \square

The set $\tilde{\mathcal{S}}_{\geq 0}^d$ can also be described as the set of all non-negative combinations (the *cone*) of rank-1 forms $\mathbf{v}\mathbf{v}^t$ with $\mathbf{v} \in \mathbb{Z}^d$:

Proposition 2.2. *We have*

$$\tilde{\mathcal{S}}_{\geq 0}^d = \mathrm{cone} \left\{ \mathbf{v}\mathbf{v}^t : \mathbf{v} \in \mathbb{Z}^d \right\}.$$

Proof. By the definition of $\tilde{\mathcal{S}}_{\geq 0}^d$, every $Q \in \tilde{\mathcal{S}}_{\geq 0}^d$ is arithmetical equivalent to a form \bar{Q} as in (2). The PQF Q' is a sum of rank-1 forms (cf. [Vor07, Section 24]). Therefore \bar{Q} and Q are of this form as well.

Suppose on the other hand that $Q \in \mathcal{S}_{\geq 0}^d$ is the sum of rank 1-forms, e.g. $Q = \sum_{i=1}^m \alpha_i \mathbf{v}_i \mathbf{v}_i^t$ with $\mathbf{v}_i \in \mathbb{Z}^d$ and $\alpha_i > 0$ for $i = 1, \dots, m$. If $\mathrm{rank} Q = d$ there is nothing to show. If $\mathrm{rank} Q < d$, then there exist linearly independent $\mathbf{p}_1, \dots, \mathbf{p}_k \in \mathbb{R}^d$ with $k = d - \mathrm{rank} Q$ such that $Q[\mathbf{p}_j] = \sum_{i=1}^m \alpha_i (\mathbf{p}_j^t \mathbf{v}_i)^2 = 0$ for $j = 1, \dots, k$. Thus the \mathbf{p}_j are orthogonal to each of the \mathbf{v}_i and therefore they span a k -dimensional linear subspace which contains a k -dimensional sublattice of \mathbb{Z}^d . We choose a basis $(\mathbf{u}_1, \dots, \mathbf{u}_k)$ of this sublattice and extend it to a basis $U = (\mathbf{u}_1, \dots, \mathbf{u}_d)$ of \mathbb{Z}^d . Then $(U^{-1})^t Q (U^{-1})$ is of the desired form (2). \square

The set $\mathrm{Del}(Q)$ of all Delone polyhedra of a $Q \in \tilde{\mathcal{S}}_{\geq 0}^d$ is called the *Delone subdivision* of Q . If all elements in $\mathrm{Del}(Q)$ are simplices, $\mathrm{Del}(Q)$ is called a *triangulation*. The subdivision $\mathrm{Del}(Q)$ is a polyhedral subdivision of \mathbb{R}^d which is invariant under translations of the form $\mathbf{x} \mapsto \mathbf{x} + \mathbf{v}$, where $\mathbf{v} \in \mathbb{Z}^d$. Therefore $\mathrm{Del}(Q)$ is completely determined by the *stars* of the translation vertices \mathbf{t}_i^t in (1) for $i = 1, \dots, m$, where a star of a single vertex is the set of all Delone polyhedra containing it. We call two Delone polyhedra P and P' *equivalent* if there exists a $\mathbf{v} \in \mathbb{Z}^d$ so that $P = \mathbf{v} + P'$. We say that $\mathrm{Del}(Q)$ is a *refinement* of $\mathrm{Del}(Q')$ (and $\mathrm{Del}(Q')$ is a *coarsening* of $\mathrm{Del}(Q)$), if every Delone polytope of Q' is contained in a Delone polytope of Q . In Section 7 we need an algorithm which computes the Delone subdivision of a given PQF. The interested reader can find a discussion of these computational issues in our paper [DSV07].

2.3. Voronoi's reduction theory. Before we generalize Voronoi's reduction theory in the next section, we briefly recall the original theory (see [Vor08], [Del37] and [SV06]). Generally, the task of reduction is to find a fundamental domain in $\mathcal{S}_{\geq 0}^d$ with

respect to the action of $\mathrm{GL}_d(\mathbb{Z})$. Voronoi's reduction is based on *secondary cones*, also called *L-type domains*, of Delone triangulations with vertex-set \mathbb{Z}^d . More generally, the *secondary cone* $\Delta(\mathcal{D})$ of a Delone subdivision \mathcal{D} with a standard periodic vertex-set is defined by

$$\Delta(\mathcal{D}) = \{Q \in \tilde{\mathcal{S}}_{\geq 0}^d : \mathrm{Del}(Q) = \mathcal{D}\}.$$

We say that two secondary cones of Delone triangulations are *bistellar neighbors* if the Delone triangulations differ by a *bistellar flip*, which is a specific change of the triangulation (see Section 3.3 for a definition and generalization). Voronoi also showed that the topological closures $\overline{\Delta(\mathcal{D})}$ of secondary cones of Delone triangulations form a polyhedral subdivision of $\tilde{\mathcal{S}}_{\geq 0}^d$.

Theorem 2.3 (*Voronoi's Reduction Theory*).

The secondary cone of a Delone triangulation with vertex-set \mathbb{Z}^d is a full-dimensional, open polyhedral cone in $\mathcal{S}_{>0}^d$. The topological closures $\overline{\Delta(\mathcal{D})}$ give a polyhedral subdivision of $\tilde{\mathcal{S}}_{\geq 0}^d$. The closures of two secondary cones have a common facet if and only if they are bistellar neighbors. The group $\mathrm{GL}_d(\mathbb{Z})$ acts on the tiling by $U \mapsto U^t \overline{\Delta(\mathcal{D})} U$. Under this group action there are only finitely many inequivalent secondary cones.

Note that by Voronoi's theory we have a non-intersecting subdivision of $\mathcal{S}_{>0}^d$, as well as of $\tilde{\mathcal{S}}_{\geq 0}^d$, into secondary cones. In it, every cone is an open polyhedral cone with respect to its affine hull. We refer to such a decomposition of $\mathcal{S}_{>0}^d$ as an *open polyhedral subdivision*. Such subdivisions are of particular interest in Section 4, if they fall into only finitely many orbits under the action of $\mathrm{GL}_d(\mathbb{Z})$, as in the case of secondary cones.

3. GENERALIZATION OF VORONOI'S REDUCTION THEORY

In this section we generalize Voronoi's reduction theory. For our generalization we consider a linear subspace $T \subseteq \mathcal{S}^d$ and look at *T-secondary cones* of Delone subdivisions \mathcal{D} defined by

$$\Delta_T(\mathcal{D}) = \Delta(\mathcal{D}) \cap T.$$

We call a *T-secondary cone* $\Delta_T(\mathcal{D})$ and the corresponding Delone subdivision \mathcal{D} *T-generic* if $\dim \Delta_T(\mathcal{D}) = \dim T$. By Voronoi's Theorem 2.3, the topological closures of *T-generic*, *T-secondary cones* give a polyhedral subdivision of $\tilde{\mathcal{S}}_{\geq 0}^d \cap T$. Two *T-generic* cones are called *contiguous* if their closures share a facet. A difference with the classical theory is the existence of *dead-ends*, which are facets only incident to one *T-generic* cone. These necessarily contain only non-positive forms in $\tilde{\mathcal{S}}_{\geq 0}^d \setminus \mathcal{S}_{>0}^d$.

The ultimate goal would be to state for every subspace T a theorem as Theorem 2.3, which deals with the case $T = \mathcal{S}^d$. It turns out though that in general, this is not always possible. If $\dim T = 1$, the intersection of T with $\mathcal{S}_{>0}^d$ contains a PQF Q and all its multiples. In this case a generalized Theorem 2.3 is trivially true. Therefore, if not stated otherwise we assume $\dim T \geq 2$ in what follows.

The “road map” for our generalization is the following: In Section 3.1 and 3.2 we determine the secondary cone of an *arbitrary* Delone subdivision *explicitly*, because in our more general setup we have to deal with Delone subdivisions which are not Delone triangulations. Then in Section 3.3 we generalize the notion of bistellar neighbors to

the new setting. In the new theory not every subspace T gives a polyhedral subdivision of $\tilde{S}_{\geq 0}^d \cap T$ with only finitely many inequivalent T -secondary cones (see Remark 4.1). For specific vertex-sets and subspaces though, there exist only finitely many inequivalent T -secondary cones. Such a finiteness result is given in Section 4 for \mathbb{Z}^d and subspaces containing all PQFs which are invariant under a given finite subgroup of $\mathrm{GL}_d(\mathbb{Z})$. Thus we obtain an equivariant version of Voronoi's reduction theory. Using modern terminology, our proofs not only generalize Voronoi's theory, but also shorten his argumentation.

3.1. Polyhedral description of secondary cones. Let \mathcal{D} be a Delone subdivision with a standard periodic vertex-set Λ . In this section we want to describe the secondary cone $\Delta(\mathcal{D})$. It will turn out that $\Delta(\mathcal{D})$ forms a (relative) open polyhedral cone in $\tilde{S}_{\geq 0}^d$. Theorem 3.1 gives the precise statement. This result is an adaption of Voronoi's "fundamental theorem" (see [Vor08, §77]), which deals with the generic case of Delone triangulations with respect to the vertex-set \mathbb{Z}^d (see also [SV06, Section 5.1]). Actually the first statement of the second part of Theorem 3.1 is not explicitly stated in Voronoi's work. It goes back to Nakamura [Nak75, Lemma 1.1].

We describe below the polyhedral cone explicitly by linear equalities and inequalities. The description is needed for our application and therefore we put some effort into avoiding redundancies. The linear equalities are coming from d -dimensional non-simplicial polyhedra in \mathcal{D} . Hence, in the generic case of Delone triangulations there are no linear equalities. The linear inequalities are coming from $(d-1)$ -dimensional polyhedra (facets) in \mathcal{D} . For the formulation of these linear conditions we define for an affinely independent set $V \subseteq \mathbb{R}^d$ of cardinality $d+1$ and a point $w \in \mathbb{R}^d$ the quadratic form

$$(3) \quad N_{V,w} = ww^t - \sum_{v \in V} \alpha_v vv^t,$$

where the coefficients α_v are uniquely determined by the affine dependency $w = \sum_{v \in V} \alpha_v v$ with $1 = \sum_{v \in V} \alpha_v$.

The following theorem generalizes Voronoi's "fundamental theorem" for Delone triangulations to arbitrary polyhedral subdivisions. As vertex-sets $\Lambda \subset \mathbb{R}^d$ we allow standard periodic sets. Moreover, we allow degenerate (unbounded) polyhedra. The *vertex-set* of a polyhedron P , denoted by $\mathrm{vert} P$, is defined as the set $\Lambda \cap P$.

Theorem 3.1. *I. Let \mathcal{D} be a polyhedral subdivision of \mathbb{R}^d with a standard periodic vertex-set Λ . Then the closure $\overline{\Delta(\mathcal{D})}$ is a polyhedral cone in $\tilde{S}_{\geq 0}^d$ and $\Delta(\mathcal{D})$ is the set of all $Q \in \tilde{S}_{\geq 0}^d$ satisfying*

(a) *for every d -dimensional polyhedron $P \in \mathcal{D}$ the equalities*

$$\langle N_{V,w}, Q \rangle = 0,$$

for one (which can be chosen arbitrarily) affinely independent set of $d+1$ vertices $V \subseteq \mathrm{vert} P$ and all $w \in \mathrm{vert} P$;

(b) *for every $(d-1)$ -dimensional polyhedron $F \in \mathcal{D}$ the inequality*

$$\langle N_{V \cup \{w\}, w'}, Q \rangle > 0,$$

for one (which can be chosen arbitrarily) affinely independent set of d vertices $V \subseteq \mathrm{vert} F$ and two vertices $w \in \mathrm{vert} P \setminus F$ and $w' \in \mathrm{vert} P' \setminus$

F of the two adjacent d -dimensional polyhedra $P, P' \in \mathcal{D}$ with $F = P \cap P'$.

- II. The map $\mathcal{D} \mapsto \overline{\Delta(\mathcal{D})}$ gives an isomorphism between the poset of Delone subdivisions of Λ ordered by coarsening and the poset of closures of secondary cones ordered by inclusion. The closures of all secondary cones of Delone subdivisions form a polyhedral subdivision of $\tilde{\mathcal{S}}_{\geq 0}^d$.

Note that different choices of V , w and w' for the inequalities could yield different conditions, respectively different forms $N_{V \cup \{w\}, w'}$. Nevertheless, these are the same on the linear subspace U defined by the equalities. In other words, their orthogonal projections $\pi_U(N_{V \cup \{w\}, w'})$ onto U are all positive multiples of a uniquely determined form $N_{\mathcal{D}, F} \in U$ with $\langle N_{\mathcal{D}, F}, N_{\mathcal{D}, F} \rangle = 1$.

Note also that for every $v \in \mathbb{R}^d$ we have

$$(4) \quad N_{V+v, w+v} = N_{V, w}.$$

Therefore, and because the vertex-set of the subdivision is assumed to be periodic, the theorem gives only finitely many inequalities. This shows that $\overline{\Delta(\mathcal{D})}$ is a polyhedral cone.

Finally, let us remark that the theorem is valid for arbitrary periodic sets, that is, finite unions of lattice translates $t_i + L$, if we replace $\tilde{\mathcal{S}}_{\geq 0}^d$ by $(A^{-1})^t \tilde{\mathcal{S}}_{\geq 0}^d (A^{-1})$ where $A \in \text{GL}_d(\mathbb{R})$ defines the lattice $L = A\mathbb{Z}^d$.

3.2. Proof of the fundamental theorem. In this section we prove Theorem 3.1. We first give two propositions which both deal with the redundancies in the set of equations and inequalities, one would obtain by considering all possible choices of V , w and w' . Proposition 3.2 takes care of the equalities and Proposition 3.5 of the inequalities. The latter shows that the orthogonal projections of all the forms $N_{V \cup \{w\}, w'}$ for a facet F , onto the linear subspace defined by the equalities, are unique up to positive multiples.

Proposition 3.2. *Let $Q \in \mathcal{S}^d$ and $V \subset \mathbb{R}^d$ be an affinely independent set of cardinality $d+1$. Let $c \in \mathbb{R}^d$ and $r > 0$ be such that $Q[c - v] = r^2$ for all $v \in V$. Then*

$$Q[w - c] - r^2 = \langle Q, N_{V, w} \rangle.$$

Proof. The proof is straightforward. We have

$$\begin{aligned} Q[w - c] - r^2 &= \langle Q, ww^t \rangle + \langle Q, -2wc^t + cc^t \rangle - r^2 \\ &= \langle Q, ww^t \rangle + \sum_{v \in V} \alpha_v \langle Q, -2vc^t + cc^t \rangle - r^2, \end{aligned}$$

with α_v as in (3). For each $v \in V$ we use the equality $Q[v - c] = r^2$ which is equivalent to $-\langle Q, vv^t \rangle = \langle Q, -2vc^t + cc^t \rangle - r^2$. This yields the desired expression. \square

Remark 3.3. By Proposition 3.2 the sign of $\langle Q, N_{V, w} \rangle$ has the following interpretation: If it is positive, then w lies outside the circumsphere of the points in V , where the circumsphere is taken with respect to the norm induced by Q . If the sign is 0, then w lies on the circumsphere, and if it is negative, then w lies inside the circumsphere. In computational geometry this insphere/outsphere test is conveniently formulated using oriented matroid terminology (cf. [BVS⁺99, Chapter 1.8]): Let $V = (v_1, \dots, v_{d+1})$

be affinely independent points in \mathbb{R}^d with positive orientation and let $\mathbf{w} \in \mathbb{R}^d$. Then the *chirotope*

$$\chi_{(\mathbf{v}_1, \dots, \mathbf{v}_{d+1}, \mathbf{w})}(Q) = \text{sign} \begin{vmatrix} 1 & \dots & 1 & 1 \\ \mathbf{v}_1 & \dots & \mathbf{v}_{d+1} & \mathbf{w} \\ Q[\mathbf{v}_1] & \dots & Q[\mathbf{v}_{d+1}] & Q[\mathbf{w}] \end{vmatrix}$$

satisfies $\chi_{(\mathbf{v}_1, \dots, \mathbf{v}_{d+1}, \mathbf{w})}(Q) = \text{sign}\langle Q, N_{V, \mathbf{w}} \rangle$.

Remark 3.4. Voronoi's theory and his description of secondary cones of Delone triangulations is based on linear forms $\varrho_{(L, L')}$ on \mathcal{S}^d , called *regulators*. Voronoi defines them for pairs of adjacent simplices (L, L') sharing a facet in a Delone triangulation. Let \mathbf{w}' be the vertex of L' which is not a vertex of L . As in Proposition 3.2, let V denote the vertex-set of L and define $N_{V, \mathbf{w}'}$. Then Voronoi's regulator $\varrho_{(L, L')}$ is a positive multiple of $\langle N_{V, \mathbf{w}'}, \cdot \rangle$.

Proposition 3.5. *Let P be a d -dimensional polyhedron in \mathbb{R}^d . Let U be the linear subspace of all $Q \in \mathcal{S}^d$ satisfying $\langle N_{V, \mathbf{w}}, Q \rangle = 0$ for all affinely independent sets $V \subseteq \text{vert } P$ of cardinality $d+1$ and for all $\mathbf{w} \in \text{vert } P$. Let F be a facet of P .*

- (1) *Let V and V' be two sets of cardinality d containing affinely independent vertices of F and let $\mathbf{w} \in \text{vert } P \setminus F$ and $\mathbf{w}' \in \mathbb{R}^d$. Then*

$$\pi_U(N_{V \cup \{\mathbf{w}\}, \mathbf{w}'}) = \pi_U(N_{V' \cup \{\mathbf{w}\}, \mathbf{w}'}).$$

- (2) *Let V be a set of cardinality d containing affinely independent vertices of F and let $\mathbf{u}, \mathbf{w} \in \text{vert } P \setminus F$ and $\mathbf{w}' \in \mathbb{R}^d$. Then*

$$\pi_U(N_{V \cup \{\mathbf{w}\}, \mathbf{w}'}) = \pi_U(N_{V \cup \{\mathbf{u}\}, \mathbf{w}'}).$$

Proof. In both cases we will show that the difference of the two considered forms lies in the orthogonal complement of U .

- (1) Every pair of affinely independent sets $V, V' \subseteq \text{vert } F$ of cardinality d can be connected by a chain $V = V_1, \dots, V_n = V'$ of affinely independent sets $V_i \subseteq \text{vert } F$ of cardinality d such that $|V_i \cap V_{i+1}| = d-1$. So we can assume $|V \cap V'| = d-1$. Setting $\alpha_{\mathbf{w}'} = 1$ there exist unique numbers $\alpha_{\mathbf{w}}$ and $\alpha_{\mathbf{v}}$ for $\mathbf{v} \in V$ defining an affine dependency between the points \mathbf{w}, \mathbf{w}' and \mathbf{v} in V . This defines the form $N_{V \cup \{\mathbf{w}\}, \mathbf{w}'}$.

We define \mathbf{v}_1 by $\mathbf{v}_1 \in V \setminus V'$. Since the affine hull of V and of V' equals the affine hull of F there exist numbers $\beta_{\mathbf{v}}$ for $\mathbf{v} \in V'$ such that $\sum_{\mathbf{v} \in V'} \beta_{\mathbf{v}} = 1$ and $\mathbf{v}_1 = \sum_{\mathbf{v} \in V'} \beta_{\mathbf{v}} \mathbf{v}$. Thus we have an affine dependency

$$\begin{cases} 0 &= \alpha_{\mathbf{w}} + \alpha_{\mathbf{w}'} + \sum_{\mathbf{v} \in V \setminus \{\mathbf{v}_1\}} \alpha_{\mathbf{v}} + \sum_{\mathbf{v} \in V'} \alpha_{\mathbf{v}_1} \beta_{\mathbf{v}}, \\ \mathbf{0} &= \alpha_{\mathbf{w}} \mathbf{w} + \alpha_{\mathbf{w}'} \mathbf{w}' + \sum_{\mathbf{v} \in V \setminus \{\mathbf{v}_1\}} \alpha_{\mathbf{v}} \mathbf{v} + \sum_{\mathbf{v} \in V'} \alpha_{\mathbf{v}_1} \beta_{\mathbf{v}} \mathbf{v} \end{cases},$$

which defines the form $N_{V' \cup \{\mathbf{w}\}, \mathbf{w}'}$. This gives

$$N_{V \cup \{\mathbf{w}\}, \mathbf{w}'} - N_{V' \cup \{\mathbf{w}\}, \mathbf{w}'} = \alpha_{\mathbf{v}_1} \left(\mathbf{v}_1 \mathbf{v}_1^t - \sum_{\mathbf{v} \in V'} \beta_{\mathbf{v}} \mathbf{v} \mathbf{v}^t \right).$$

Since $V \cup V'$ is a minimal affinely dependent set, we can choose an arbitrary vertex \mathbf{w} of P and find that the right hand side is a multiple of $N_{V' \cup \{\mathbf{w}\}, \mathbf{v}_1}$.

- (2) We take a closer look at the difference $N_{V \cup \{w\}, w'} - N_{V \cup \{u\}, w'}$ and show it is a multiple of $N_{V \cup \{w\}, u}$. Set $\alpha_{w'} = 1$ again and let α_w and α_v with $v \in V$ be real numbers defining the affine dependency between the points w, w' and v in V . This defines the form $N_{V \cup \{w\}, w'}$. In the same way let $\alpha'_{w'} = 1, \alpha'_u$ and α'_v be real numbers defining $N_{V \cup \{u\}, w'}$.

We set $\beta_u = \alpha'_u, \beta_w = -\alpha_w$ and $\beta_v = \alpha'_v - \alpha_v$ for $v \in V$. Then $\sum \beta_v v = 0$ and $\sum \beta_v = 0$ where the sums run through all $v \in V \cup \{u, w\}$. Thus

$$(5) \quad N_{V \cup \{w\}, w'} - N_{V \cup \{u\}, w'} = \beta_u N_{V \cup \{w\}, u}.$$

□

Note, if u and w' in the last calculation lie in opposite halfspaces with respect to the affine plane through V , then $\beta_u > 0$ in (5). Therefore, repeated application yields the following proposition, which we use for the proof of Theorem 3.1 and in Section 3.3 to prove Theorem 3.8.

Proposition 3.6. *Let $V_1, \dots, V_m \subset \mathbb{R}^d$ be affinely independent sets of cardinality $d+1$ with $|V_i \cap V_{i+1}| = d$ for $i = 1, \dots, m-1$. Let $w \in \mathbb{R}^d$ and V_i be on opposite sides of $\text{aff}(V_i \cap V_{i+1})$ for $i = 1, \dots, m-1$. Then*

$$N_{V_1, w} = N_{V_m, w} + \sum_{i=1}^{m-1} \alpha_i N_{V_i, v_{i+1}}$$

with $v_{i+1} \in V_{i+1} \setminus V_i$ and positive constants α_i , for $i = 1, \dots, m-1$.

With these propositions at hand, we can give a proof of the “fundamental theorem”.

Proof of Theorem 3.1. I. We show that $\Delta(\mathcal{D})$ is given by the set of listed linear equalities and inequalities. By (4) this implies that $\overline{\Delta(\mathcal{D})}$ is a polyhedral cone, because the Delone subdivision induced by a $Q \in \tilde{\mathcal{S}}_{\geq 0}^d$ and a standard periodic vertex-set contains only finitely many Delone polyhedra up to \mathbb{Z}^d invariant translations.

For $Q \in \Delta(\mathcal{D})$ the linear equalities and inequalities are satisfied by Proposition 3.2.

Conversely, let us assume $Q \in \mathcal{S}^d$ satisfies the linear equalities and inequalities for every polytope in \mathcal{D} . By Proposition 3.2 and Proposition 3.5 we can assume that these are valid for all possible choices of V, v and w . Note that for the use of Proposition 3.5 it is crucial to observe, that for two linear subspaces U, U' of \mathcal{S}^d with $T = U \cap U'$ we have $\pi_T = \pi_U \circ \pi_{U'} = \pi_{U'} \circ \pi_U$.

Let P be a d -dimensional polyhedron in \mathcal{D} . Let $V \subseteq \text{vert } P$ be the vertex-set of a d -simplex.

We show that all possible inequalities $\langle N_{V, w}, Q \rangle \geq 0$, where $w \in \mathcal{D}$, are implied by the inequalities $\langle N_{V, v}, Q \rangle \geq 0$, where $v \in \mathcal{D}$ is either a vertex of P or a vertex of an adjacent Delone polyhedron of P . Assume $w \in \mathcal{D} \setminus P$. Then we choose a sequence of adjacent d -simplices with vertex-sets V_1, \dots, V_m in \mathcal{D} satisfying the requirements of Proposition 3.6. In addition we require that each vertex-set is contained in the vertex-set of a fixed polyhedron of \mathcal{D} , in particular $V_1 \subseteq \text{vert } P$ and $w \in V_m$. Note that this can be achieved by looking at a refining triangulation of \mathcal{D} , which we can choose arbitrarily. By our assumption on Q , we have $\langle N_{V_i, v_{i+1}}, Q \rangle \geq 0$ with equality

if and only if V_i, V_{i+1} are subsets of the same polyhedron. Thus by Proposition 3.6 and Proposition 3.2 we see that the equation $\langle N_{V_m, \mathbf{w}}, Q \rangle = 0$ implies $\langle N_{V_1, \mathbf{w}}, Q \rangle > 0$.

Now we prove that a form $Q \in \mathcal{S}^d$ satisfying all linear conditions must be positive semidefinite. Let $\mathbf{w} \in \mathcal{D}$ and $\lambda \in \mathbb{Z}$ and write $\lambda \mathbf{w} = \sum_{\mathbf{v} \in V} \alpha_{\mathbf{v}} \mathbf{v}$ with $1 = \sum_{\mathbf{v} \in V} \alpha_{\mathbf{v}}$. Then

$$0 \leq \langle N_{V, \lambda \mathbf{w}}, Q \rangle = \lambda^2 Q[\mathbf{w}] - \sum_{\mathbf{v} \in V} \alpha_{\mathbf{v}} Q[\mathbf{v}],$$

and hence $Q[\mathbf{w}] \geq 0$ since the coefficients $\alpha_{\mathbf{v}}$ depend only affinely on λ .

Furthermore, Q lies in the rational closure $\tilde{\mathcal{S}}_{\geq 0}^d$: The linear conditions are rational and thus define a polyhedral cone in which every extreme ray contains rational semidefinite quadratic forms. By the argument in the proof of Proposition 2.2 all quadratic forms in such a polyhedral cone lie in $\tilde{\mathcal{S}}_{\geq 0}^d$.

We finally show that there exists a $\mathbf{c} \in \mathbb{R}^d$ and $r > 0$ such that $Q[\mathbf{w} - \mathbf{c}] \geq r^2$ with equality if and only if $\mathbf{w} \in \text{vert } P$. A solution of the system of linear equations $Q[\mathbf{v} - \mathbf{c}] = Q[\mathbf{v}' - \mathbf{c}]$, where $\mathbf{v}, \mathbf{v}' \in V$ with $V \subseteq \text{vert } P$, gives \mathbf{c} and r . Applying Proposition 3.2 shows that the inequalities $Q[\mathbf{w} - \mathbf{c}] \geq r^2$ are valid with equality if and only if $\mathbf{w} \in \text{vert } P$.

II. For the assertion on the isomorphism of posets given by the map $\mathcal{D} \mapsto \overline{\Delta(\mathcal{D})}$, we need to verify that the Delone subdivision \mathcal{D} is a true coarsening of \mathcal{D}' if and only if $\overline{\Delta(\mathcal{D})}$ is strictly contained in $\overline{\Delta(\mathcal{D}')}$. This follows from I. because if \mathcal{D} is a true coarsening of \mathcal{D}' we have equalities in the description of $\Delta(\mathcal{D})$ which are inequalities in the description of $\Delta(\mathcal{D}')$. If on the other hand, $\overline{\Delta(\mathcal{D})}$ is strictly contained in $\overline{\Delta(\mathcal{D}')}$, we know that all equalities and inequalities in the description of $\Delta(\mathcal{D}')$ are also satisfied by elements of $\Delta(\mathcal{D})$ implying that \mathcal{D} is a coarsening of \mathcal{D}' . Moreover, $\overline{\Delta(\mathcal{D})}$ has to be contained in the boundary of $\overline{\Delta(\mathcal{D}')}$, because otherwise there would exist a $Q \in \Delta(\mathcal{D}') \cap \Delta(\mathcal{D})$ implying $\mathcal{D} = \mathcal{D}'$. Thus at least one of the inequalities in the description of $\Delta(\mathcal{D}')$ is fulfilled with equality for the elements of $\overline{\Delta(\mathcal{D})}$. Hence \mathcal{D} is a *true* coarsening of \mathcal{D}' .

Since every form in $\tilde{\mathcal{S}}_{\geq 0}^d$ defines a unique Delone subdivision of Λ , $\tilde{\mathcal{S}}_{\geq 0}^d$ is subdivided into secondary cones. This subdivision has the face-to-face property, hence is a polyhedral subdivision, due to the poset isomorphism: Otherwise we would find Delone subdivisions \mathcal{D} and \mathcal{D}' , such that $\overline{\Delta(\mathcal{D})}$ and $\overline{\Delta(\mathcal{D}')}$ have the same dimension and such that there is a common relative interior point, but with $\overline{\Delta(\mathcal{D})} \neq \overline{\Delta(\mathcal{D}')}$. Thus Q would yield two different Delone subdivisions, which is not possible. \square

3.3. Flips and bistellar neighbors. Given a linear subspace T of \mathcal{S}^d , we know by Theorem 3.1 that $\tilde{\mathcal{S}}_{\geq 0}^d \cap T$ is covered by the topological closures of T -generic T -secondary cones. As before we assume that the vertex-set of the considered Delone subdivisions is a fixed standard periodic set Λ . Given a T -generic Delone subdivision \mathcal{D} and its T -secondary cone $\Delta_T(\mathcal{D})$ we want to determine its contiguous T -secondary cones. For this let \mathcal{F} be a facet of $\Delta_T(\mathcal{D})$ which is not a dead-end. Let \mathcal{D}' be the T -generic Delone subdivision with $\Delta(\mathcal{D}')$ being contiguous to $\Delta(\mathcal{D})$ at \mathcal{F} , hence with $\mathcal{F} = \overline{\Delta_T(\mathcal{D})} \cap \overline{\Delta_T(\mathcal{D}')}$. The transition from \mathcal{D} to \mathcal{D}' is called a *T-flip*; the subdivisions \mathcal{D} and \mathcal{D}' are referred to as *bistellar neighbors*.

In order to describe the T -flip, let $N_{\mathcal{F}} \in \mathcal{S}^d$ denote the form which is uniquely determined by the following conditions:

- (i) $N_{\mathcal{F}} \in T$ with $\langle N_{\mathcal{F}}, N_{\mathcal{F}} \rangle = 1$,
- (ii) $\langle N_{\mathcal{F}}, Q \rangle = 0$ for all $Q \in \mathcal{F}$,
- (iii) $\langle N_{\mathcal{F}}, Q \rangle > 0$ for all $Q \in \Delta_T(\mathcal{D})$,
- (iv) $\langle N_{\mathcal{F}}, Q \rangle < 0$ for all $Q \in \Delta_T(\mathcal{D}')$.

We collect facets F of \mathcal{D} whose forms $N_{\mathcal{D},F}$ (as defined after Theorem 3.1) are positive multiples of $N_{\mathcal{F}}$ when projected onto T :

$$\mathcal{R}_{\mathcal{F}} = \{F \in \mathcal{D} : \dim F = d - 1 \text{ and } N_{\mathcal{F}} = \alpha \cdot \pi_T(N_{\mathcal{D},F}) \text{ for a } \alpha > 0\}.$$

On $\mathcal{R}_{\mathcal{F}}$ we define an equivalence relation by

$$(6) \quad F \sim F' \iff \begin{cases} \text{there exist } F_1, \dots, F_n \in \mathcal{R}_{\mathcal{F}} \text{ with } F = F_1, F' = F_n, \\ \text{and } d\text{-dimensional polyhedra } P_0, \dots, P_n \in \mathcal{D} \\ \text{with } F_i = P_{i-1} \cap P_i \text{ for } i = 1, \dots, n. \end{cases}$$

Thus by definition, the facets in each equivalence class are connected by a chain of adjacent d -dimensional polyhedra in \mathcal{D} . The union of these polyhedra is a polyhedron again:

Proposition 3.7. *Let \mathcal{D} be a Delone subdivision and \mathcal{C} an equivalence class of (6). Let $V_{\mathcal{C}}$ be the set of all vertices of d -dimensional polytopes in \mathcal{D} with a facet in \mathcal{C} . Then $(\text{conv } V_{\mathcal{C}}) \cap \Lambda = V_{\mathcal{C}}$.*

A proof of the proposition is given at the end of this section. It, as well as the description of T -flips can conveniently be given using the *lifting map* $w_Q : \mathbb{R}^d \rightarrow \mathbb{R}^{d+1}$ defined by $w_Q(x) = (x, Q[x])$ for a $Q \in \tilde{\mathcal{S}}_{\geq 0}^d$. The polyhedra $\text{conv } V_{\mathcal{C}}$ in Proposition 3.7 are called *repartitioning polyhedra* of \mathcal{F} . For a $Q \in \Delta_T(\mathcal{D})$ and a repartitioning polyhedron P we define the $(d+1)$ -dimensional polyhedron

$$\bar{P}(Q) = \text{conv}\{w_Q(v) : v \in \text{vert } P\}.$$

For each of its facets \bar{F} we have an outer normal vector $\mathbf{n}(\bar{F}) \in \mathbb{R}^{d+1}$. The facets are divided into three groups. We speak of a *lower facet*, if the last coordinate of the normal vector satisfies $(\mathbf{n}(\bar{F}))_{d+1} < 0$; we speak of an *upper facet*, if $(\mathbf{n}(\bar{F}))_{d+1} > 0$ and of a *lateral facet*, if $(\mathbf{n}(\bar{F}))_{d+1} = 0$. We show below that these notions are independent of the particular choice of $Q \in \Delta_T(\mathcal{D})$.

For the description of the T -flip in the following theorem, let $\pi : \mathbb{R}^{d+1} \mapsto \mathbb{R}^d$ denote the projection onto the first d coordinates. Figure 1. gives an example for the change from “lower to upper hull” in the case of a planar repartitioning polytope. Note that the situation in higher dimensions can be much more complicated than the picture might suggest.

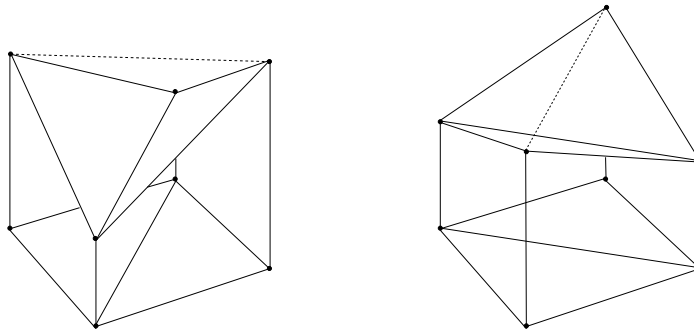


Figure 1. Lifting interpretation of “upper to lower hull change”.

Theorem 3.8. *Let T be a linear subspace of S^d and \mathcal{F} be a common facet of two contiguous T -secondary cones of T -generic Delone subdivisions \mathcal{D} and \mathcal{D}' . Then we obtain the d -dimensional polytopes of \mathcal{D}' from those of \mathcal{D} by choosing an arbitrary $Q \in \Delta(\mathcal{D})$ and by doing the following for each repartitioning polytope P of \mathcal{F} :*

- (i) *We remove all polytopes $\pi(\bar{F})$ of lower facets \bar{F} of $\bar{P}(Q)$.*
- (ii) *We add all polytopes $\pi(\bar{F})$ of upper facets \bar{F} of $\bar{P}(Q)$.*

Remark 3.9. Our theory is similar to the theory of equivariant secondary polytopes of regular polytopal subdivisions, as recently described by Reiner in [Rei02]. With slight modifications our explicit description of the flipping procedure works in his setting as well. The linear subspace T is in this case containing all PQFs which are invariant under a given finite subgroup of $\text{GL}_d(\mathbb{Z})$ (see Section 4).

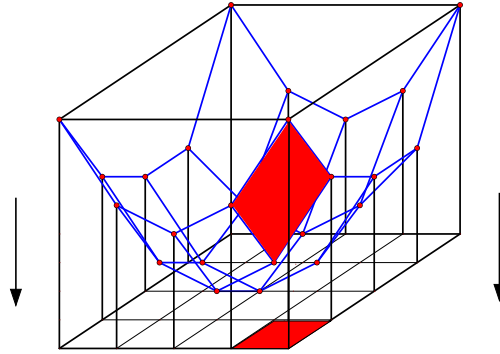
As a preparation for the proofs of Theorem 3.8 and Proposition 3.7, consider the set

$$(7) \quad \text{conv}\{w_Q(\mathbf{x}) : \mathbf{x} \in \Lambda\}.$$

It is a *locally finite polyhedron*, meaning that the intersection with a polytope (bounded polyhedron) is a polytope again. A first and important observation is that the facets of the polyhedron (7) yield the d -polytopes of the Delone subdivision when projected by π onto the first d coordinates (see Figure 2):

Proposition 3.10. *Let Λ be a standard periodic set and $Q \in \tilde{S}_{\geq 0}^d$. Then*

$$\text{Del}(Q) = \{\pi(\bar{F}) : \bar{F} \text{ face of (7) of dimension less than or equal } d\}.$$

**Figure 2.** Delone subdivision obtained from lifting.

Proof. For a polytope $P \in \mathcal{D}$ and a set $V \subseteq \text{vert } P$ of cardinality $d + 1$ and with affinely independent vertices, the set of all $(\mathbf{x}, y) \in \mathbb{R}^{d+1}$ with

$$(8) \quad y = Q[\mathbf{x}] - \langle N_{V,\mathbf{x}}, Q \rangle = \sum_{v \in V} \alpha_v Q[v]$$

is an affine hyperplane in \mathbb{R}^{d+1} . Recall from the definition of $N_{V,\mathbf{x}}$ in (3) that the α_v are uniquely defined by the affine dependency $\mathbf{x} = \sum_{v \in V} \alpha_v \mathbf{v}$ with $\sum_{v \in V} \alpha_v = 1$. Thus we see that the hyperplane given by (8) contains the $d + 1$ lifted points $w_Q(\mathbf{v})$. Moreover, it is a supporting hyperplane of the polyhedron (7) if and only if $\langle N_{V,\mathbf{x}}, Q \rangle \geq 0$

for all $x \in \Lambda$. By Proposition 3.2 this is the case if and only if $Q[x - c] \geq r^2$ for a suitable $c \in \mathbb{R}^d$ and $r > 0$. Here, equality holds if and only if $x \in \text{vert } P$. \square

The hyperplane given by equation (8) in the proof is a different one for subsets V and V' of different Delone polytopes. The forms $N_{\mathcal{D},F}$ relate the change for adjacent d -polytopes P and P' in \mathcal{D} with common facet $F = P \cap P'$. If the segment connecting x with a vertex v of P intersects facets F_1, \dots, F_n of \mathcal{D} , then by Proposition 3.6 and by the definition of $N_{\mathcal{D},F}$ (after and because of Theorem 3.1) we have

$$(9) \quad y = Q[x] - \langle N_{V,x}, Q \rangle = Q[x] - \sum_{i=1}^n \beta_i \langle N_{\mathcal{D},F_i}, Q \rangle$$

with suitable positive constants β_i , depending on x .

Equation (9) yields simple proofs for Proposition 3.7 and Theorem 3.8:

Proof of Proposition 3.7. Due to (9) and the definition of \mathcal{C} , all points $w_Q(v)$ with $v \in V_{\mathcal{C}}$ are coplanar in \mathbb{R}^{d+1} . Suppose the union of the d -dimensional polyhedra of \mathcal{D} with a facet in \mathcal{C} is strictly contained in $\text{conv } V_{\mathcal{C}}$. Then there exists a facet F of one of the polyhedra such that $F \notin \mathcal{C}$ and F does not belong to the boundary of $\text{conv } V_{\mathcal{C}}$. Hence, there exist two vertices $w, w' \in V_{\mathcal{C}}$ in opposite halfspaces with respect to $\text{aff } F$. Then by (9) and since $\langle N_{\mathcal{D},F}, Q \rangle > 0$ we see that $w_Q(w)$, $w_Q(w')$ and $w_Q(v)$ with $v \in F$ can not be coplanar, which is a contradiction. \square

Proof of Theorem 3.8. Choose $Q \in \Delta(\mathcal{D})$ and $Q' \in \Delta(\mathcal{D}')$ and let P be a repartitioning polytope of $\mathcal{F} = \overline{\Delta(\mathcal{D})} \cap \overline{\Delta(\mathcal{D}')}$. Then by formula (9) and since $N_{\mathcal{D},F} = -N_{\mathcal{D}',F}$ for all facets F contained in P , the upper facets of $\bar{P}(Q)$ project to the same polyhedral subdivision of P as the lower facets of $\bar{P}(Q')$ and vice versa. The lower facets give the corresponding Delone subdivisions by Proposition 3.10, which proves the assertion. \square

4. FINITENESS AND EQUIVARIANCE

For the sake of simplicity and because it is sufficient for our applications in Section 8 and Section 9 we restrict ourselves from now on to the case of Delone subdivisions with vertex-set \mathbb{Z}^d . A more general discussion can be found in [Sch07]. In particular it is shown that the results of this Section extend to the case of *rational standard periodic* vertex-sets $\Lambda = \bigcup_{i=1}^m t_i + \mathbb{Z}^d$ with $t_i \in \mathbb{Q}^d$ for $i = 1, \dots, m$.

Let T be a linear subspace of S^d . We say that two T -secondary cones Δ_T and Δ'_T are *T-equivalent* if there is a $g \in \text{GL}_d(\mathbb{Z})$ so that $g^t \Delta_T g = \Delta'_T$ and $g^t T g = T$. Otherwise we say that they are *T-inequivalent*. In this section we discuss assumptions on T which ensure that there exist only finitely many, T -inequivalent T -secondary cones. Note that Voronoi's classical theory (cf. Theorem 2.3) deals with the case $T = S^d$.

The main difference with the classical theory is that for general T there may be infinitely many, T -inequivalent T -secondary cones:

Example 4.1. Consider a rational PQF Q with trivial automorphism group. The orbit of Q under the action of $\text{GL}_d(\mathbb{Z})$ contains only rational PQFs. We choose a subspace T through Q in which the only rational forms are multiples of Q . So T does not contain another element from the orbit of Q . Thus the setwise stabilizer of T is trivial.

In this situation there are infinitely many T -generic secondary cones whenever the set $T \cap \tilde{\mathcal{S}}_{\geq 0}^d$ is not closed in \mathcal{S}^d . Then the intersection of T with the boundary of $\tilde{\mathcal{S}}_{\geq 0}^d$ is not covered by finitely many T -dead-ends. The maximum dimension of the intersection of $\mathcal{S}_{\geq 0}^d$ with a supporting hyperplane of $\mathcal{S}_{\geq 0}^d$ is $\binom{d+1}{2} - d$ (see [BR79, §10]). Thus the intersection of T with the boundary of $\tilde{\mathcal{S}}_{\geq 0}^d$ cannot be covered by T -dead-ends if $\dim(T \cap \mathcal{S}_{\geq 0}^d) > \binom{d+1}{2} - d + 1$. In particular for all $d \geq 3$ there exist examples of subspaces \bar{T} with infinitely many, inequivalent T -generic cones.

In the following we show that there are only finitely many, T -inequivalent T -secondary cones if T is the linear subspace which is stabilized pointwise by a finite subgroup G of $\mathrm{GL}_d(\mathbb{Z})$. We develop this equivariant theory in analogy to the theory of G -perfect forms of Bergé, Martinet and Sigrist [BMS92]. In Remark 4.4 we give a unifying view on both theories.

First let us recall some definitions and basic results (see [BNZ73, Section 2]). Let G be a finite subgroup of $\mathrm{GL}_d(\mathbb{Z})$. The linear subspace

$$\mathcal{F}(G) = \{Q \in \mathcal{S}^d : g^t Q g = Q \text{ for all } g \in G\}$$

is called the *space of invariant forms* of G . The pointwise stabilizer of $\mathcal{F}(G)$, which we denote by

$$\mathcal{B}(G) = \{g \in \mathrm{GL}_d(\mathbb{Z}) : g^t Q g = Q \text{ for all } Q \in \mathcal{F}(G)\},$$

is called the *Bravais group* of G . Note that $\mathcal{B}(G)$ can be strictly larger than G . The *normalizer* of a subgroup G of $\mathrm{GL}_d(\mathbb{Z})$ is defined by $N(G) = \{n \in \mathrm{GL}_d(\mathbb{Z}) : n^{-1} G n = G\}$. One important property of $N(\mathcal{B}(G))$ is that it is the setwise stabilizer of the subspace of invariant forms

$$N(\mathcal{B}(G)) = \{g \in \mathrm{GL}_d(\mathbb{Z}) : g^t \mathcal{F}(G) g = \mathcal{F}(G)\}.$$

A proof can be found for example in [Jaq95, Lemme 3.2].

Now we are ready to state the main result of this section. For this recall that an open polyhedral subdivision is a non-intersecting decomposition into polyhedral cones which are open with respect to their affine hull.

Theorem 4.2. *Let \mathcal{P} be an open polyhedral subdivision of $\mathcal{S}_{>0}^d$ on which $\mathrm{GL}_d(\mathbb{Z})$ acts by $(g, \Delta) \mapsto g^t \Delta g$. Suppose that this action gives only finitely many orbits. Define a polyhedral subdivision of $\mathcal{F}(G) \cap \mathcal{S}_{>0}^d$ by*

$$\mathcal{P}_{\mathcal{F}(G)} = \{\Delta \cap \mathcal{F}(G) : \Delta \in \mathcal{P}\}.$$

Then the normalizer $N(\mathcal{B}(G))$ acts on $\mathcal{P}_{\mathcal{F}(G)}$ and there exist only finitely many orbits with respect to this action.

This together with Theorem 2.3 gives the following corollary, which completes our equivariant version of Voronoi's theory.

Corollary 4.3. *Let G be a finite subgroup of $\mathrm{GL}_d(\mathbb{Z})$ and let $T = \mathcal{F}(G)$ be the space of invariant forms. Then there exist only finitely many T -inequivalent T -secondary cones of Delone subdivisions of \mathbb{Z}^d .*

Proof of Theorem 4.2. We start with a definition. For $\Delta \in \mathcal{P}$ we define the automorphism group $\mathrm{Aut}(\Delta) = \{g \in \mathrm{GL}_d(\mathbb{Z}) : g^t \Delta g = \Delta\}$.

We consider the set

$$\mathcal{O}_{\Delta,G} = \{g^t \Delta g \cap \mathcal{F}(G) : g \in \mathrm{GL}_d(\mathbb{Z}) \text{ and } g^t \Delta g \cap \mathcal{F}(G) \neq \emptyset\}.$$

The normalizer $N(\mathcal{B}(G))$ stabilizes $\mathcal{F}(G)$ setwise, and hence it acts on $\mathcal{O}_{\Delta,G}$. We show that $\mathcal{O}_{\Delta,G}$ is a finite union of $N(\mathcal{B}(G))$ -orbits, that is, there are $g_1, \dots, g_m \in \mathrm{GL}_d(\mathbb{Z})$ with

$$\mathcal{O}_{\Delta,G} = \bigcup_{i=1}^m \{n^t (g_i^t \Delta g_i \cap \mathcal{F}(G)) n : n \in N(\mathcal{B}(G))\}.$$

Then the statement of the theorem follows because the set $\{\mathcal{O}_{\Delta,G} : \Delta \in \mathcal{P}\}$ is finite by the assumption made on \mathcal{P} .

Let g_1, g_2 be in $\mathrm{GL}_d(\mathbb{Z})$ so that $g_i^t \Delta g_i \cap \mathcal{F}(G) \in \mathcal{O}_{\Delta,G}$ for $i = 1, 2$. The group $\mathcal{B}(G)$ is a subgroup of $\mathrm{Aut}(g_1^t \Delta g_1)$ and $\mathrm{Aut}(g_2^t \Delta g_2)$ which can be seen as follows. By definition the group $\mathcal{B}(G)$ stabilizes the set $g_i^t \Delta g_i \cap \mathcal{F}(G)$ pointwise. Since $\mathrm{GL}_d(\mathbb{Z})$ operates on \mathcal{P} and $g_i^t \Delta g_i \cap \mathcal{F}(G) \neq \emptyset$ every element of $\mathcal{B}(G)$ has to stabilize $g_i^t \Delta g_i$ setwise.

Hence, $g_1 \mathcal{B}(G) g_1^{-1}$ and $g_2 \mathcal{B}(G) g_2^{-1}$ are subgroups of $\mathrm{Aut}(\Delta)$. Assume that $g_1 \mathcal{B}(G) g_1^{-1} = g_2 \mathcal{B}(G) g_2^{-1}$. Then $n = g_2^{-1} g_1$ is an element of the normalizer $N(\mathcal{B}(G))$. Furthermore, we have

$$n^t (g_2^t \Delta g_2 \cap \mathcal{F}(G)) n = n^t g_2^t \Delta g_2 n \cap n^t \mathcal{F}(G) n = g_1^t \Delta g_1 \cap \mathcal{F}(G)$$

because $N(\mathcal{B}(G))$ stabilizes $\mathcal{F}(G)$ setwise. Thus different $N(\mathcal{B}(G))$ -orbits in $\mathcal{O}_{\Delta,G}$ induce different subgroups of $\mathrm{Aut}(\Delta)$.

For every $\Delta \in \mathcal{P}$ the group $\mathrm{Aut}(\Delta)$ is finite (This fact was already proved by Nakamura in [Nak75, Lemma 1.2]. For completeness we give an argument, which also is a bit more elementary than Nakamura's.): Let $\mathcal{R}_1, \dots, \mathcal{R}_k \in \mathcal{S}_{\geq 0}^d$ be the extreme rays spanning the closed polyhedral cone $\overline{\Delta}$. Choose forms $Q_i \in \mathcal{R}_i$ in the following way: If for a pair $\mathcal{R}_i, \mathcal{R}_j$ there is a $U \in \mathrm{Aut}(\Delta)$ with $U^t \mathcal{R}_i U = \mathcal{R}_j$ then $U^t Q_i U = Q_j$. We have $\overline{\Delta} = \mathrm{cone}\{Q_1, \dots, Q_k\}$ and $Q = \sum_{i=1}^k Q_i \in \Delta$ and in particular $Q \in \mathcal{S}_{>0}^d$. So, $\mathrm{Aut}(Q) = \{g \in \mathrm{GL}_d(\mathbb{Z}) : g^t Q g = Q\}$ is finite. By construction $\mathrm{Aut}(Q)$ contains $\mathrm{Aut}(\Delta)$. So, the group $\mathrm{Aut}(\Delta)$ has only a finite number of different subgroups. So, $\mathcal{O}_{\Delta,G}$ is a finite union of $N(\mathcal{B}(G))$ -orbits. \square

We close this section with two remarks.

Remark 4.4. The proof of Theorem 4.2 is essentially an adaptation of proofs of Jaquet-Chiffelle ([Jaq95, Théorème 5.2]) and Opgenorth ([Opg95, Theorem 4.2.18]). They show that there exist only finitely many inequivalent G -perfect forms (definition is given below). Proofs for their theorems can be derived from Theorem 4.2 in the following way:

Let m be a positive number and let \mathcal{P}_m be the set

$$\mathcal{P}_m = \{Q \in \mathcal{S}_{>0}^d : Q[v] \geq m \text{ for all } v \in \mathbb{Z}^d \setminus \{0\}\}.$$

This is a convex, locally finite polyhedral cone. For each face F of \mathcal{P}_m we define the (relatively) open polyhedral cone $\Delta_F = \{\lambda Q : \lambda > 0 \text{ and } Q \in \mathrm{relint} F\}$. Then $\mathcal{P} = \{\Delta_F : F \text{ face of } \mathcal{P}_m\}$ gives an open polyhedral subdivision of $\mathcal{S}_{>0}^d$ as required in Theorem 4.2.

If G is a finite subgroup of $\mathrm{GL}_d(\mathbb{Z})$, then Q is called G -perfect if for the cone Δ_F with $Q \in \Delta_F$ we have $\dim(\Delta_F \cap \mathcal{F}(G)) = 1$. Two G -perfect forms Q and Q' are called G -equivalent if there is a $g \in N(\mathcal{B}(G))$ and a positive λ so that $g^t Q g = \lambda Q'$. Otherwise they are called G -inequivalent. By Voronoi's first memoir [Vor07] the polyhedral subdivision \mathcal{P} satisfies the assertion of Theorem 4.2. Hence, there are only finitely many G -inequivalent G -perfect forms by our Theorem 4.2.

Usually G -perfect forms (cf. [Mar03, BMS92, Jaq95, Opg95, Opg01]) are defined via *normal cones* of faces F_G of $\mathcal{P}_m \cap \mathcal{F}(G)$ in $\mathcal{F}(G)$. A face F of \mathcal{P}_m is uniquely characterized by the set

$$\mathrm{Min}(F) = \{v \in \mathbb{Z}^d : Q[v] = m \text{ for all } Q \in F\},$$

which is independent of the particular choice of m . The normal cone $\mathcal{N}(F)$ of F is given by $\mathcal{N}(F) = \mathrm{cone}\{vv^t : v \in \mathrm{Min}(F)\}$ and the normal cone of the face $F_G = F \cap \mathcal{F}(G)$ in $\mathcal{F}(G)$ is obtained by an orthogonal projection of $\mathcal{N}(F)$ onto $\mathcal{F}(G)$. If different inner products are used, the resulting cones differ as seen in the cases of [Jaq95] and [Opg95].

Remark 4.5. The case when the linear subspace T is pointwise stabilized by a finite subgroup of $\mathrm{GL}_d(\mathbb{Z})$ is not the only case when there are only finitely many T -inequivalent T -secondary cones. Bayer–Fluckiger and Nebe ([BN05, Theorem 3.1]) give another sufficient condition for T which ensures finiteness in the two-dimensional case. However, we are not aware of further results regarding this question. For all the subspaces T we considered for the applications, there were only finitely many T -inequivalent T -secondary cones. These subspaces are all spanned by rational forms, but this alone does not guarantee finiteness for $d \geq 3$, as shown by the following example due to Yves Benoist (private communication):

Consider the subspace T spanned by $x^2 + 2y^2 + z^2$ and xy . It contains the two positive semidefinite, but non-positive forms $(x \pm \sqrt{2}y)^2 + z^2$. In any small neighborhood of these two forms, we find infinitely many forms in $T \cap S_{>0}^d$ with pairwise differing Delone subdivisions. In particular, the minimal non-zero vectors $v \in \mathbb{Z}^d$ tend towards the kernel of one of the two forms; the Delone subdivisions have differing edges $\mathrm{conv}\{\mathbf{0}, v\}$ with $v = (p, q, 0)^t$ and $\frac{p}{q}$ sufficiently close to $\mp\sqrt{2}$. Thus there are infinitely many different T -secondary cones, but the setwise stabilizer of T in $\mathrm{GL}_d(\mathbb{Z})$ is finite. In order to see this, note that the three lines of rank-2 forms in T have to be permuted. Note that the same argument does not apply to dimension 2. The space spanned by $x^2 + 2y^2$ and xy has an infinite stabilizer.

5. OTHER GENERALIZATIONS OF VORONOI'S REDUCTION THEORY

In the seventies, Mumford and Satake described a general procedure for compactifying quotients of bounded symmetric domains by arithmetic groups (cf. [AMRT75]). Their constructions use special decompositions of the domains of interest into rational polyhedra. As an application of this general method, Namikawa ([Nam76]) constructed a compactification of Siegel modular varieties using Voronoi's reduction theory. Siegel modular varieties are moduli spaces of Abelian varieties. Over the complex numbers they arise as quotient of the Siegel upper halfspace by an arithmetic group. Namikawa's construction opened the possibility to understand the boundary of the moduli space with help of degenerations. Later algebraic geometers refined this work

and they used also generalizations of Voronoi's reduction theory for this. In this section we want to review these generalizations and compare them to ours. For more information about the geometry of Siegel modular varieties we refer to the survey [HS02] of Hulek and Sankaran.

Oda and Seshadri study in [OS79] a generalization of Delone subdivisions which they call Namikawa decompositions: Let E be a Euclidean space where an orthogonal decomposition into subspaces E' and E'' is given: $E = E' \perp E''$, and let $\Lambda \subseteq E$ be a lattice so that $\Lambda \cap E'$ is also a lattice. For a vector $\psi \in E''$ the Namikawa decomposition is

$$D_\psi = \{\pi'(D(x)) : x \in E' + \psi\},$$

where $\pi' : E \rightarrow E'$ is the orthogonal projection onto E' and

$$D(x) = \text{conv}\{\xi_1, \dots, \xi_r\},$$

with vertices $\xi_i \in \Lambda$ closest to x , is a Delone polytope of Λ . Then, Oda and Seshadri explain in [OS79, Proposition 2.3] how the Namikawa decomposition changes when one varies the vector ψ along E'' . Our generalization of Voronoi's reduction theory goes into a different direction: We fix a lattice Λ in a real vector space of finite dimension and a basis of Λ and study how the Delone subdivision of Λ changes when one varies the inner product along a given subspace (in the space of Gram matrices).

Alexeev and collaborators (cf. [ABH02, Ale02, Ale04] and the survey [Ale06, Section 5]) consider so-called semi-Delaunay decompositions: Let $M \subseteq \mathbb{R}^d$ be a lattice and $\Gamma \subseteq M$ be a sublattice of finite index. A Γ -periodic polyhedral subdivision of \mathbb{R}^d is called a semi-Delaunay decomposition when it is the projection of the lower hull (see Section 3.3) of the lifted points $(m, h(m))$ with $m \in M$ and $h : M \rightarrow \mathbb{R}$ a function of the form $h(m) = q(m) + r(m)$ where $q : \mathbb{R}^d \rightarrow \mathbb{R}$ is a positive semidefinite quadratic form and $r : M/\Gamma \rightarrow \mathbb{R}$ an arbitrary function on the cosets. Delone subdivisions of *rational standard periodic sets*, i.e. sets of the form (1) with rational t'_i , are covered by this construction: Take $\Gamma = \mathbb{Z}^d$ and an overlattice M containing all t'_i . Define $r(t'_i + \Gamma) = 0$ and $r(m + \Gamma)$ large enough for all $m \in M$ not in the rational standard periodic set.

Nevertheless, we believe that our treatment has its merits since it is very explicit and can be immediately used for computations. Moreover, our generalization of Voronoi's reduction theory to positive definite quadratic forms with prescribed automorphism group, which is essential for our applications, is not covered by the reviewed constructions.

6. OPTIMIZING LATTICE SPHERE COVERINGS

The lattice covering problem is a classical problem in "Geometry of Numbers". Roughly speaking, the problem is concerned with the determination of the most economical way to cover \mathbb{R}^d . We deal with it in Section 8 and 9 where we apply our generalization of Voronoi's theory. In this section we give the necessary definitions and describe briefly the methods used for the applications. For further reading and more background we refer the interested reader to [SV06].

6.1. Definitions. We assume that L is a lattice of full rank d , that is, there exists a matrix $B \in \text{GL}_d(\mathbb{R})$ with $L = B\mathbb{Z}^d$. The *determinant* $\det(L) = |\det(B)| > 0$ of L is well defined and does not depend on the chosen basis. Let $B^d = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$

denote the solid unit sphere in \mathbb{R}^d . The Minkowski sum $L + \alpha B^d = \{\mathbf{v} + \alpha \mathbf{x} : \mathbf{v} \in L, \mathbf{x} \in B^d\}$, with $\alpha \in \mathbb{R}_{>0}$, is called a *lattice covering* if $\mathbb{R}^d = L + \alpha B^d$. The *covering radius* $\mu(L)$ of L is given by

$$\mu(L) = \min\{\mu : L + \mu B^d \text{ is a lattice covering}\}.$$

For $\alpha \in \mathbb{R}$ we have

$$\mu(\alpha L) = |\alpha| \mu(L)$$

Thus, the *covering density*

$$\Theta(L) = \frac{\mu(L)^d}{\det(L)} \cdot \kappa_d,$$

with $\kappa_d = \text{vol } B^d$, is invariant with respect to scaling of L . Note also that Θ is an invariant of the isometry classes.

The *lattice covering problem* asks to minimize Θ among all d -dimensional lattices. It has been solved only for dimensions $d \leq 5$ based on Voronoi's classical reduction theory and the knowledge of all Delone subdivisions in these dimensions.

If we work with PQFs, the definitions above translate in the following way:

$$\Theta(Q) = \Theta(L) = \sqrt{\frac{\mu(Q)^d}{\det Q}} \cdot \kappa_d,$$

Here, the *inhomogeneous minimum* $\mu(Q)$ is given by

$$\mu(Q) = \max_{\mathbf{x} \in \mathbb{R}^d} \min_{\mathbf{v} \in \mathbb{Z}^d} Q[\mathbf{x} - \mathbf{v}],$$

satisfying $\det(L) = \sqrt{\det(Q)}$, $\mu(L) = \sqrt{\mu(Q)}$, for a corresponding lattice L obtained from Q .

Given a linear subspace $T \subseteq \mathcal{S}^d$, let

$$\Theta_T = \inf_{Q \in T \cap \mathcal{S}_{>0}^d} \Theta(Q)$$

denote the bound on the covering density with respect to T . This bound is computed in our applications described in Section 8 and Section 9.

6.2. Determinant maximization problems. Let $T \subseteq \mathcal{S}^d$ be a linear subspace. To compute Θ_T , we consider all inequivalent T -generic Delone subdivision \mathcal{D} (if possible) and minimize Θ among all PQFs in $\overline{\Delta}_T(\mathcal{D})$. This can be achieved by solving a *determinant maximization problem*. These are convex programming problems of the form

$$(10) \quad \begin{array}{ll} \text{minimize} & \mathbf{c}^t \mathbf{x} - \log \det G(\mathbf{x}) \\ \text{subject to} & G(\mathbf{x}) \succ 0, F(\mathbf{x}) \succeq 0, \end{array}$$

over the variable vector $\mathbf{x} \in \mathbb{R}^D$. The objective function contains a linear part given by $\mathbf{c} \in \mathbb{R}^D$. The affine maps $G : \mathbb{R}^D \rightarrow \mathbb{R}^{m \times m}$, as well as $F : \mathbb{R}^D \rightarrow \mathbb{R}^{n \times n}$, are given by

$$\begin{aligned} G(\mathbf{x}) &= G_0 + x_1 G_1 + \cdots + x_D G_D, \\ F(\mathbf{x}) &= F_0 + x_1 F_1 + \cdots + x_D F_D, \end{aligned}$$

where $G_i \in \mathbb{R}^{m \times m}$ and $F_i \in \mathbb{R}^{n \times n}$, $i = 0, \dots, D$, are symmetric matrices. The notation $G(\mathbf{x}) \succ 0$ and $F(\mathbf{x}) \succeq 0$ gives the constraints “ $G(\mathbf{x})$ is positive definite”

and “ $F(\mathbf{x})$ is positive semidefinite”. Note that we specialize to a *semidefinite programming problem*, if $G(\mathbf{x})$ is the identity matrix for all $\mathbf{x} \in \mathbb{R}^D$. One nice feature of determinant maximization problems is that there is a duality theory similar to the one of linear programming (see [VBW98]). It allows us to compute an interval in which the optimum is attained, the so called *duality gap*.

We can express the condition $\mu(Q) \leq 1$ as a *linear matrix inequality (LMI)* $F(\mathbf{x}) \succeq 0$ where the optimization vector \mathbf{x} is given by the coefficients of $Q = \sum_{i=1}^{\dim T} x_i A_i$, with respect to some basis $(A_1, \dots, A_{\dim T})$ of T . This is seen by the following proposition due to Delone et al [DDRS70] (cf. [SV06, Proposition 7.1]), together with the crucial observation that the inner product (\cdot, \cdot) defined by $(\mathbf{y}, \mathbf{z}) = \mathbf{y}^t Q \mathbf{z}$ can be expressed as

$$(\mathbf{y}, \mathbf{z}) = \langle Q, \mathbf{y} \mathbf{z}^t \rangle = \sum_{i=1}^{\dim T} x_i \langle A_i, \mathbf{y} \mathbf{z}^t \rangle,$$

hence as a linear combination of the parameters x_i .

Proposition 6.1. *Let $L = \text{conv}\{\mathbf{0}, \mathbf{v}_1, \dots, \mathbf{v}_d\} \subseteq \mathbb{R}^d$ be a d -dimensional simplex. Then the circumradius of L is at most 1 with respect to (\cdot, \cdot) if and only if*

$$\text{BR}_L(Q) = \begin{pmatrix} 4 & (\mathbf{v}_1, \mathbf{v}_1) & (\mathbf{v}_2, \mathbf{v}_2) & \dots & (\mathbf{v}_d, \mathbf{v}_d) \\ (\mathbf{v}_1, \mathbf{v}_1) & (\mathbf{v}_1, \mathbf{v}_1) & (\mathbf{v}_1, \mathbf{v}_2) & \dots & (\mathbf{v}_1, \mathbf{v}_d) \\ (\mathbf{v}_2, \mathbf{v}_2) & (\mathbf{v}_2, \mathbf{v}_1) & (\mathbf{v}_2, \mathbf{v}_2) & \dots & (\mathbf{v}_2, \mathbf{v}_d) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\mathbf{v}_d, \mathbf{v}_d) & (\mathbf{v}_d, \mathbf{v}_1) & (\mathbf{v}_d, \mathbf{v}_2) & \dots & (\mathbf{v}_d, \mathbf{v}_d) \end{pmatrix} \succeq 0.$$

Since a block matrix is semidefinite if and only if the blocks are semidefinite, we get the desired result. Here, we call two Delone polytopes P and P' *equivalent with respect to $Q \in \mathcal{S}_{>0}^d$* , if there exists a U in the *automorphism group*

$$\text{Aut}(Q) = \{U \in \text{GL}_d(\mathbb{Z}) : U^t Q U = Q\}$$

of Q and a $\mathbf{v} \in \mathbb{Z}^d$ with $P = \mathbf{v} + U P'$.

Proposition 6.2. *Let $Q \in \mathcal{S}_{>0}^d$ be a PQF with Delone subdivision \mathcal{D} . Let P_1, \dots, P_n be a representative system of d -dimensional Delone polytopes in \mathcal{D} , which are inequivalent with respect to Q . For every P_i choose a d -dimensional simplex L_i with $\text{vert } L_i \subseteq \text{vert } P_i$. Then*

$$\mu(Q) \leq 1 \iff \begin{pmatrix} \boxed{\text{BR}_{L_1}(Q)} & 0 & 0 & \dots & 0 \\ 0 & \boxed{\text{BR}_{L_2}(Q)} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \boxed{\text{BR}_{L_n}(Q)} \end{pmatrix} \succeq 0.$$

Thus the constraint $\mu(Q) \leq 1$ can be brought into one LMI of type $F(\mathbf{x}) \succeq 0$. We can add linear constraints on the parameters x_i by extending F by a 1×1 block matrix for each linear inequality. In this way we obtain one LMI for the two constraints $\mu(Q) \leq 1$ and $Q \in \overline{\Delta}(\mathcal{D})$. Therefore we can determine a PQF in $\overline{\Delta}_T(\mathcal{D})$ minimizing the covering density by solving the following determinant maximization problem.

$$\begin{aligned} &\textbf{minimize} && -\log \det(Q) \\ &\textbf{subject to} && Q \in \overline{\Delta}_T(\mathcal{D}), \mu(Q) \leq 1. \end{aligned}$$

Here $F(\mathbf{x})$ is as described above, and $\mathbf{c} = \mathbf{0}$, and $G(\mathbf{x}) = \sum_{i=1}^{\dim T} x_i A_i$ in (10) where the A_i form a basis of T .

Remark 6.3. The problem is a convex programming problem. Hence (see e.g. [SV06, Proposition 9.1]), if the automorphism group of the considered Delone subdivision \mathcal{D} contains the pointwise stabilizer of the subspace T , we know that a PQF minimizing the covering density in $\overline{\Delta(\mathcal{D})}$ is contained in T . Hence by choosing the subspace T carefully we sometimes can dramatically decrease the dimension of the optimization problem.

Remark 6.4. Determinant maximization problems have not only theoretically nice features, but also can be solved in practice quite well, for example with interior point methods. For the solution of the optimization problems described above we use the software package MAXDET [WVB96] of Wu, Vandenberghe, and Boyd as a subroutine. By using rational approximations it is possible to give mathematical rigorous error bounds or even proofs of local optimality. For details we refer to [SV06]. Utilizing these ideas we developed a program `rmc` (*rigorous maxdet*) and based on it `coop`, a *covering optimizer*. These programs, together with a short tutorial can be obtained from our web page [SV05a]. They allow to approximate — mathematically rigorously — the optimum covering density with respect to a given T -secondary cone.

7. ALGORITHMIC ISSUES

In this section we discuss several algorithmic issues which have to be resolved to turn our generalization of Voronoi's reduction theory into an effective procedure. As in Section 4 we assume that all Delone subdivisions have \mathbb{Z}^d as a vertex-set. For the original theory of Voronoi a similar discussion can be found in [SV06, Section 5.3]. Here we emphasize those points where the generalization differs from the classical case. First we adapt [SV06, Algorithm 1], which enumerates all inequivalent Delone triangulations, to an algorithm enumerating all T -inequivalent T -generic Delone subdivisions. As in Example 4.1, our Algorithm 1 below does not necessarily stop after finitely many steps, depending on T .

Apart from this, when we compare Algorithm 1 to the one of the original theory, essentially two new tasks arise:

- (1) We have to find an initial T -generic Delone subdivision (cf. Algorithm 2), whereas in the classical case this can be given by “Voronoi's first subdivision”.
- (2) We have to check whether two T -generic Delone subdivisions are T -equivalent (cf. Algorithm 3).

For Algorithm 2 we need one more definition: The dimension of the linear span of $\Delta_T(\mathcal{D})$ is called T -*rigidity index*. For the classical setting $T = \mathcal{S}^d$ the *rigidity index* was introduced by Baranovskii and Grishukhin in [BG01]. Algorithm 2 is a randomized algorithm and belongs to the class of so-called Las Vegas algorithms. That is, it always produces correct results whereas the running time is a random variable.

Algorithm 3 checks whether two T -generic T -secondary cones are T -equivalent. For this we need a definition.

Definition 7.1. We say that a positive semidefinite quadratic form Q is *rational normalized* if it is integral and the entries have greatest common divisor equal to one.

Input: Linear subspace $T \subseteq \mathcal{S}^d$
Output: Set \mathcal{R} of all T -inequivalent T -generic Delone subdivisions
 $Y \leftarrow \{\mathcal{D}_1\}$, where \mathcal{D}_1 is a T -generic Delone subdivision (Algorithm 2).
 $\mathcal{R} \leftarrow \emptyset$.
while there is a $\mathcal{D} \in Y$ **do**
 $Y \leftarrow Y \setminus \{\mathcal{D}\}$. $\mathcal{R} \leftarrow \mathcal{R} \cup \{\mathcal{D}\}$.
 Compute the linear inequalities of $\overline{\Delta_T(\mathcal{D})}$ as in Theorem 3.1.
 Compute the facets F_1, \dots, F_n of $\overline{\Delta_T(\mathcal{D})}$.
 for $i = 1, \dots, n$ **do**
 if F_i is not a dead-end **then**
 Compute the bistellar neighbor \mathcal{D}_i of \mathcal{D} , defined by F_i as in Theorem 3.8.
 if \mathcal{D}_i is not T -equivalent to a Delone subdivision in the set
 $\mathcal{R} \cup \{\mathcal{D}_j : j \in \{1, \dots, i-1\}, F_j \text{ not a dead-end}\}$ (Algorithm 3) **then**
 $Y \leftarrow Y \cup \{\mathcal{D}_i\}$.
 end if
 end if
 end for
end while

Algorithm 1. Enumeration of all T -inequivalent T -generic Delone subdivisions.

Input: Linear subspace $T \subseteq \mathcal{S}^d$
Output: A T -generic Delone subdivision \mathcal{D}_1
repeat
 Choose a random Q in $T \cap \mathcal{S}_{>0}^d$.
 Compute $\mathcal{D}_1 = \text{Del}(Q)$ (see [DSV07]).
 Compute the T -rigidity index m of \mathcal{D}_1 by the linear equalities in Theorem 3.1.
until $m = \dim T$.

Algorithm 2. Finding a T -generic Delone subdivision.

For Algorithm 3 we have to test whether two PQFs are equivalent and we have to compute the automorphism group of a given PQF. For both tasks exist efficient algorithms by Plesken and Souvignier [PS97]. Implementations are part of the computational algebra system MAGMA and of the computer package CARAT (Crystallographic Algorithms And Tables, cf. [OPS98]).

The closed polyhedral cones $\overline{\Delta_T(\mathcal{D})}$ are *rational* (assuming that the vertex-set is \mathbb{Z}^d). That is, they have a description by rational inequalities (cf. Section 3) as well as a description $\overline{\Delta_T(\mathcal{D})} = \text{cone}\{R_1, \dots, R_k\}$ as a cone generated by finitely many rational normalized quadratic forms lying in extreme rays of $\overline{\Delta_T(\mathcal{D})}$.

Dealing with symmetries of a polyhedral cone, the *characteristic form* $Q = \sum_{i=1}^k R_i$ is of great importance. This is due to the following proposition which is used in Algorithm 3. We call two rational polyhedral cones \mathcal{P} and \mathcal{P}' in \mathcal{S}^d *equivalent*, if there exists a $U \in \text{GL}_d(\mathbb{Z})$ such that $U^t \mathcal{P} U = \mathcal{P}'$. As in the proof of Theorem 4.2, the *automorphism group* of \mathcal{P} is defined by $\text{Aut}(\mathcal{P}) = \{U \in \text{GL}_d(\mathbb{Z}) : U^t \mathcal{P} U = \mathcal{P}\}$.

Input: Linear subspace $T \subseteq S^d$, and T -generic Delone subdivisions $\mathcal{D}_1, \mathcal{D}_2$
Output: Yes, if \mathcal{D}_1 and \mathcal{D}_2 are T -equivalent; No, otherwise
 $\mathcal{R}_i \leftarrow \{\text{rational normalized element } R \text{ of extreme rays of } \overline{\Delta_T(\mathcal{D}_i)}\}, \text{ for } i = 1, 2.$
 $Q_i \leftarrow \sum_{R \in \mathcal{R}_i} R, \text{ for } i = 1, 2.$
 Compute $\text{Aut}(Q_1)$.
if there exists $A \in \text{GL}_d(\mathbb{Z})$ and $B \in \text{Aut}(Q_1)$
 with $A^t Q_1 A = Q_2$ and $(BA)^t T (BA) = T$ **then**
 return Yes.
else
 return No.
end if

Algorithm 3. Checking T -equivalence.

Lemma 7.2. *The automorphism group of a characteristic form contains the automorphism group of its polyhedral cone. The characteristic forms of two equivalent rational polyhedral cones are equivalent.*

Proof. Suppose that for two polyhedral cones $\mathcal{P}, \mathcal{P}' \subset \tilde{S}_{\geq 0}^d$ there exists a $U \in \text{GL}_d(\mathbb{Z})$ such that $U^t \mathcal{P} U = \mathcal{P}'$. Extreme rays of \mathcal{P} are mapped onto extreme rays of \mathcal{P}' . We need to show that this is also true for the uniquely determined rational normalized elements of the extreme rays. For this let Q be such a rational normalized element of an extreme ray of \mathcal{P} . It is mapped onto the integral matrix $U^t Q U = \alpha Q'$ defining an extreme ray of \mathcal{P}' , with rational normalized Q' . Thus $\alpha \in \mathbb{N}$. On the other hand Q' is mapped onto an integral multiple of Q via U^{-1} and $(U^{-1})^t Q' (U^{-1}) = (1/\alpha)Q$. Thus $\alpha = 1$, which proves the assertion. \square

8. APPLICATION I: CLASSIFICATION OF TOTALLY REAL THIN NUMBER FIELDS

In this section we apply our theory to a problem in algebraic number theory. Recently, Bayer–Fluckiger introduced in [Bay06] the notion of thin algebraic number fields. A thin algebraic number field is Euclidean due to a special geometric reason, which we explain in Section 8.1. She proved that there exist only finitely many thin number fields ([Bay06, Proposition 11.4]).

In [BN05] Bayer–Fluckiger and Nebe gave a complete list ([BN05, Theorem 5.1]) of 17 candidates for totally real thin algebraic number fields. They proved that 13 of them are thin and one of them is weakly thin but not thin. However, in three cases they did not know how to decide whether the fields are thin or not. Using our theory we finish the classification of totally real thin algebraic number fields. In particular we show in Section 8.2 that the three open cases do not give weakly thin fields.

8.1. Background and definitions. Let us recall some standard definitions from algebraic number theory.

Let K be a number field of degree $n = [K : \mathbb{Q}]$. From now on we assume that K is *totally real*, that is, for all embeddings of fields $\sigma_i : K \rightarrow \mathbb{C}$ with $i = 1, \dots, n$ we have $\sigma_i(K) \subseteq \mathbb{R}$.

Let \mathfrak{o}_K be its ring of integers which is a \mathbb{Z} -module of rank n . Thus we can write $\mathfrak{o}_K = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$. We embed \mathfrak{o}_K into \mathbb{R}^n via the map σ defined by $\sigma(x) = (\sigma_1(x), \dots, \sigma_n(x))$. Hence, $\sigma(\mathfrak{o}_K)$ is a lattice of rank n .

By \mathcal{P} we denote the set of $\alpha \in K$ with $\sigma_i(\alpha) > 0$ for all $i = 1, \dots, n$. Then

$$\langle x, y \rangle_\alpha = \sum_{i=1}^n \sigma_i(\alpha xy) = \sum_{i=1}^n \sigma_i(\alpha) \sigma_i(x) \sigma_i(y)$$

defines an inner product on K and so on \mathbb{R}^n . Every inner product $\langle \cdot, \cdot \rangle : K \times K \rightarrow \mathbb{R}$ with the additional property $\langle x, yz \rangle = \langle xz, y \rangle$ for all $x, y, z \in K$ is of this form.

We denote by $\Lambda_{K,\alpha}$ the lattice which is given by the pair $(\mathfrak{o}_K, \langle \cdot, \cdot \rangle_\alpha)$ where $\alpha \in \mathcal{P}$. We say that K is *weakly thin* if for $\Theta(K) = \min_{\alpha \in \mathcal{P}} \Theta(\Lambda_{K,\alpha})$ we have the inequality

$$\Theta(K) \leq \sqrt{\frac{n^n}{(\det \Lambda_{K,1})^2}} \cdot \kappa_n,$$

and we say that K is *thin* if we have strict inequality. By $t(K)$ we denote the number on the right hand side of this inequality.

Let us briefly explain the motivation of this definition. First note that $(\det \Lambda_{K,1})^2$ equals the discriminant d_K of K . Let $N : K \rightarrow \mathbb{R}$ be the norm of K which is given by $N(x) = \prod_{i=1}^n \sigma_i(x)$. A number field K is called *Euclidean* if its ring of integers \mathfrak{o}_K is an Euclidean ring with respect to the absolute value of the norm function. Equivalently, K is Euclidean if and only if its Euclidean minimum

$$M(K) = \sup_{x \in K} \inf_{y \in \mathfrak{o}_K} |N(x - y)|$$

is strictly less than 1. Currently there is no algorithm known which computes the Euclidean minimum of a number field. It is also an open problem if there exists a finite or an infinite number of Euclidean number fields. In [Len77] Lenstra, motivated by work of Hurwitz, gave several bounds on the Euclidean minimum of a number field using methods from “Geometry of Numbers”. In [Bay06] Bayer-Fluckiger extended these results. By using the inequality between the arithmetic and geometric means to relate the norm function N to the norm $\|x\|_\alpha = \sqrt{\langle x, x \rangle_\alpha} = \sqrt{\sum_{i=1}^n \sigma_i(\alpha x^2)}$, she showed that thin fields are Euclidean fields ([Bay06, Proposition 11.2]) and that there exist only finitely many thin fields ([Bay06, Proposition 11.4]).

8.2. Classification. By applying the lower bound for sphere coverings of Coxeter, Few and Rogers [CFR59], Bayer-Fluckiger showed that the degree of a totally real thin number field is at most 5 ([Bay06, Proof of Proposition 11.4]). All number fields having low degree and low discriminant are known ([PH05]). Using this list Bayer-Fluckiger and Nebe gave a complete list of 17 candidates of totally real number fields which might be thin. In Table 1 we list these candidates K together with the relevant parameters: The degree n , the discriminant d_K , the bound $t(K)$ we defined above and the minimum covering density $\Theta(K)$.

Bayer-Fluckiger and Nebe showed by giving a number α , which defines the appropriate inner product $\langle \cdot, \cdot \rangle_\alpha$, that 13 candidates are thin. For computing an upper bound of $\Theta(K)$ as given in the table we used their values $\alpha \in \mathcal{P}$ in all but one case. In the case of $\mathbb{Q}[x]/(x^3 + x^2 - 3x - 1)$ their table contains a misprint. Instead of $\alpha = 1 - 18\bar{x} + 10\bar{x}^2$ which does not lie in \mathcal{P} we use $\alpha = 2 - \bar{x}$ instead. For one candidate Bayer-Fluckiger

n	d_K	K	$t(K)$	$\Theta(K)$	thin
2	5	$\mathbb{Q}[x]/(x^2 - 5)$	≥ 2.8099	≤ 1.2645	yes
	8	$\mathbb{Q}[x]/(x^2 - 2)$	≥ 2.2214	≤ 1.3463	yes
	12	$\mathbb{Q}[x]/(x^2 - 3)$	≥ 1.8137	≤ 1.2092	yes
	13	$\mathbb{Q}[x]/(x^2 - 13)$	≥ 1.7426	≤ 1.5708	yes
	17	$\mathbb{Q}[x]/(x^2 - 17)$	≥ 1.5238	≤ 1.2497	yes
	21	$\mathbb{Q}[x]/(x^2 - 21)$	≥ 1.3711	≤ 1.2242	yes
	24	$\mathbb{Q}[x]/(x^2 - 6)$	≥ 1.2825	≤ 1.2583	yes
3	49	$\mathbb{Q}[x]/(x^3 + x^2 - 2x - 1)$	≥ 3.1093	≤ 1.5584	yes
	81	$\mathbb{Q}[x]/(x^3 - 3x + 1)$	≥ 2.4183	≤ 2.1225	yes
	148	$\mathbb{Q}[x]/(x^3 + x^2 - 3x - 1)$	≥ 1.7891	≤ 1.7014	yes
	169	$\mathbb{Q}[x]/(x^3 + x^2 - 4x + 1)$	≤ 1.6743	≥ 1.8544	no
4	725	$\mathbb{Q}[x]/(x^4 - x^3 - 3x^2 + x + 1)$	≥ 2.9323	≤ 2.7045	yes
	1125	$\mathbb{Q}[x]/(x^4 - x^3 - 4x^2 + 4x + 1)$	≥ 2.3540	≤ 2.2935	yes
	1600	$\mathbb{Q}[x]/(x^4 + 2x^3 - 5x^2 - 6x - 1)$	≤ 1.9740	≥ 2.2853	no
	1957	$\mathbb{Q}[x]/(x^4 - 4x^2 - x + 1)$	≤ 1.7849	≥ 1.8939	no
	2000	$\mathbb{Q}[x]/(x^4 - 5x^2 + 5)$	$= \Theta_4$	$= \Theta_4$	weakly
5	14641	$\mathbb{Q}[x]/(x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1)$	≥ 2.4318	≤ 2.2961	yes

Table 1. Classification of totally real thin number fields.

and Nebe showed that it is weakly thin. In this case $t(K) = \Theta(K) = \Theta_4$ where Θ_4 is the least lattice covering density among all 4-dimensional lattices. It is uniquely attained by the lattice A_4^* .

The other three cases were left open by Bayer–Fluckiger and Nebe and by our computation it turns out that they do not give thin fields. Generally, for a totally real number field K , there exists a subspace T of S^n whose dimension equals the degree of K , and such that $\Theta(K) = \Theta_T$. The corresponding subspace T is given by the basis $(\langle \omega_i, \omega_j \rangle_{\alpha_k})_{1 \leq i, j \leq n}$, $k = 1, \dots, n$, where $\mathfrak{o}_K = \mathbb{Z}\omega_1 + \dots + \mathbb{Z}\omega_n$ and $\alpha_1, \dots, \alpha_n$ forms a \mathbb{Q} -basis of K with $\alpha_k \in \mathcal{P}$. Thus if there exist only finitely many inequivalent T -secondary cones, we can compute $\Theta(K) = \Theta_T$ by the methods explained in Section 6.

For the field $K = \mathbb{Q}[x]/(x^3 + x^2 - 4x + 1)$ we have 7 generic T -secondary cones and for all lattices $\Lambda_{K, \alpha}$ our computation proves the bound $\Theta(\Lambda_{K, \alpha}) \geq 1.8544$. For the field $K = \mathbb{Q}[x]/(x^4 + 2x^3 - 5x^2 - 6x - 1)$ we have 47 generic T -secondary cones and for all lattices $\Lambda_{K, \alpha}$ our computation shows $\Theta(\Lambda_{K, \alpha}) \geq 2.2853$. For the field $K = \mathbb{Q}[x]/(x^4 - 4x^2 - x + 1)$ we have 341 generic T -secondary cones and for all lattices $\Lambda_{K, \alpha}$ our computation gives $\Theta(\Lambda_{K, \alpha}) \geq 1.8939$.

The fact that in all these cases the number of inequivalent T -generic secondary cones is finite comes as a pleasant surprise. In this situation T is not the space of invariant forms of a Bravais group so that we can not a priori rely on the finiteness result of Section 4. It remains an interesting open problem to give a proof of this fact, which does not rely on the complete enumeration of T -secondary cones.

d	lattice	covering density Θ	d	lattice	covering density Θ
1	\mathbb{Z}^1	1	13	L_{13}^c	7.762108
2	A_2^*	1.209199	14	L_{14}^c	8.825210
3	A_3^*	1.463505	15	L_{15}^c	11.004951
4	A_4^*	1.765529	16	A_{16}^*	15.310927
5	A_5^*	2.124286	17	A_{17}^9	12.357468
6	L_6^c	2.464801	18	A_{18}^*	21.840949
7	L_7^c	2.900024	19	A_{19}^{10}	21.229200
8	L_8^c	3.142202	20	A_{20}^7	20.366828
9	L_9^c	4.268575	21	A_{21}^{11}	27.773140
10	L_{10}^c	5.154463	22	Λ_{22}^*	≤ 27.8839
11	L_{11}^c	5.505591	23	Λ_{23}^*	≤ 15.3218
12	L_{12}^c	7.465518	24	Λ_{24}	7.903536

Table 2. Least dense known (lattice) coverings up to dimension 24.

9. APPLICATION II: NEW BEST KNOWN SPHERE COVERINGS

In this section we explain how the new theory can be used to construct good and even new best known sphere coverings. With the described methods, we were in particular able to construct new best known (lattice) sphere coverings in dimensions $9, \dots, 15$. Table 2 gives an overview on the currently best known sphere coverings up to dimension 24. This table is an update of the table given in [CS88, Table 2.1]. Note in particular, that in comparison to the table there, we have new best known sphere coverings in all dimensions $d \in \{6, \dots, 21\} \setminus \{16, 18\}$. Note that the problem has been solved for $d \leq 5$ only, by the work of Ryshkov and Baranovskii [RB75]. We keep an updated list with additional informations on the involved lattices on our web page [SV05a].

This section is organized as follows: First we give some background information on the lattices of Table 2. Then the next two sections deal with two different constructions of subspaces T we used. Both contain specific information on how we obtained the new best known lattices. The last section gives some information on a similar approach to the closely related packing-covering problem.

9.1. Some background on best known lattice coverings. The new sphere coverings in dimension 6, 7 and 8 were obtained and described in detail in [SV06] and [SV05b]. The lattices A_d^r , where r divides $d + 1$, are the *Coxeter lattices* [Cox51]. A possible definition is via the root lattice

$$A_d = \{\mathbf{x} \in \mathbb{Z}^{d+1} : \sum_{i=0}^{d+1} x_i = 0\}.$$

The Coxeter lattice is the lattice generated by A_d and the vector

$$(1/r) \left(\sum_{i=1}^{d+1} \mathbf{e}_i \right) - (\mathbf{e}_1 + \dots + \mathbf{e}_{(d+1)/r}),$$

where e_i denotes the i -th standard basis vector. It is the unique sublattice of A_d^* containing A_d as a sublattice of index r . In particular $A_d^* = A_d^{d+1}$. Other well known lattices in the series are $A_7^2 = E_7$, $A_7^4 = E_7^*$ and $A_8^3 = E_8$.

Since the symmetric group S_{d+1} acts on the lattice A_d^r , it is possible (with the help of a computer) to enumerate all its orbits of Delone polytopes and to compute their covering densities in fairly large dimensions. A detailed description of the method together with results up to dimension 27 can be found in [DSV07].

Baranovskii [Bar94] computed (by hand) the Delone decomposition of A_9^5 , finding the former best known sphere covering in dimension 9. Anzin [Anz02] computed the Delone decomposition of A_{11}^4 and A_{13}^7 establishing the former covering records in those dimensions. In a private communication he reported on computing the covering densities of A_{14}^5 and A_{15}^8 , which were also best known ones in their dimension. Hence suitable Coxeter lattices provide good covering lattices. Nevertheless, by applying our theory, we found that all of the five mentioned lattices do not even give a locally optimal lattice covering. However, these lattices give good starting points. In fact, we obtained the new covering records in dimensions $d = 9, 11, 13, 14, 15$ by applying our theory to a suitable linear subspace T containing a PQF of the corresponding Coxeter lattices. More details are given in the next section. For dimensions $d = 10, 12$ we used a lamination technique, which is described thereafter.

The entries of Table 2 for dimensions $d = 22, 23, 24$ are “consequences” of the existence of the Leech lattice Λ_{24} . It may not surprise that the Leech lattice itself yields the best known lattice covering in dimension 24. Its covering density was computed by Conway, Parker and Sloane ([CS88, Chapter 23]). In [SV05b] it is shown that the Leech lattice gives at least a local optimum of the covering function $\Theta(Q)$. Note that the root lattice E_8 does not have this property. Knowing the comparatively low covering density of the Leech lattice, Smith [Smi88] was able to estimate the covering densities of the *dual laminated lattices* Λ_{22}^* and Λ_{23}^* .

9.2. Large subgroups and small linear subspaces. One strategy to find good covering lattices is to consider subspaces T containing a PQF, which gives a good or even best known sphere covering. In order to keep the number of T -secondary cones low (manageable), one preferably chooses a subspace T of low dimension, e.g. less than or equal to 4. Another problem is the size of Delone subdivisions to be dealt with. If T is contained in a space of invariant forms $\mathcal{F}(G)$ of a preferably large finite subgroup G of $GL_d(\mathbb{Z})$, then this problem can be reduced by exploiting the symmetries of the Delone subdivisions, respectively those of the forms $Q \in \mathcal{F}(G)$ (see Proposition 6.2).

Dimension 9. We optimized over a 3-dimensional subspace T containing a PQF of the former record lattice A_9^5 . We computed 210 T -generic T -secondary cones. The new covering record is attained by a PQF which has 34 orbits of Delone polytopes.

Dimension 11. We optimized over a 3-dimensional subspace T containing a PQF of the former record lattice A_{11}^4 . We computed 2444 T -generic T -secondary cones. The new covering record is attained by a PQF which has 99 orbits of Delone polytopes.

Dimension 13. We optimized over a 2-dimensional subspace T containing a PQF of the former record lattice A_{13}^7 . We computed 79 T -generic T -secondary cones. The new covering record is attained by a PQF which has 134 orbits of Delone polytopes.

Dimension 14. We optimized over a 2-dimensional subspace T containing a PQF of the former record lattice A_{14}^5 . We computed 162 T -generic T -secondary cones. The new covering record is attained by a PQF which has 983 orbits of Delone polytopes.

Dimension 15. We optimized over a 2-dimensional subspace T containing a PQF of the former record lattice A_{15}^8 . We computed 109 T -generic T -secondary cones. The new covering record is attained by a PQF which has 203 orbits of Delone polytopes.

9.3. Best coverings from laminations. Another fruitful strategy is the construction of good coverings from lower dimensional ones. Assume that $Q \in \mathcal{S}_{>0}^d$ is contained in a linear subspace T of \mathcal{S}^d . We consider a corresponding lattice $L = A\mathbb{Z}^d$ with $A \in \text{GL}_d(\mathbb{R})$. Further, we choose a point $A\mathbf{c}$ with $\mathbf{c} \in \mathbb{R}^d$ and consider all lattices L_λ in \mathbb{R}^{d+1} generated by

$$A' = \begin{pmatrix} A & A\mathbf{c} \\ 0 & \lambda \end{pmatrix} \in \text{GL}_{d+1}(\mathbb{R})$$

with $\lambda > 0$ and associated PQF

$$Q' = \begin{pmatrix} Q & Q\mathbf{c} \\ \mathbf{c}^t Q & \mathbf{c}^t Q\mathbf{c} + \lambda^2 \end{pmatrix} \in \mathcal{S}_{>0}^{d+1}.$$

Choosing Q within T and $\lambda^2 \in \mathbb{R}$, we obtain a linear subspace T' of forms Q' with $\dim T' = \dim T + 1$. If $A\mathbf{c}$ is the center of a Delone polytope P of the lattice L , then the lattice vectors in the lattice L_λ belonging to the i -th layer

$$\left\{ \begin{pmatrix} A & A\mathbf{c} \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ i \end{pmatrix} : \mathbf{v} \in \mathbb{Z}^d \right\}, \quad \text{with } i \in \mathbb{Z},$$

have at least distance $|i|\lambda$ from $(A\mathbf{c}, 0)$. Hence, P is a Delone polytope embedded in L_λ when λ is at least the circumradius of P in L .

Dimension 10. We considered the 1-dimensional linear space T containing A_9^5 . We look at the linear subspace T' of \mathcal{S}^{10} of dimension 2 constructed with the center of a Delone polytope having the largest symmetry group in A_9^5 . We computed 6 T' -generic T' -secondary cones. The new covering record is attained by a PQF which has 4 orbits of Delone polytopes.

Dimension 12. We considered the 1-dimensional linear space T containing A_{11}^4 . We look at the linear subspace T' of \mathcal{S}^{12} of dimension 2 constructed with the center of a Delone polytope. We computed 241 T' -generic T' -secondary cones. The new covering record is attained by a PQF which has 1206 orbits of Delone polytopes. Here we tried all different centers. The new covering record was produced by using a center of a Delone polytope having the third largest symmetry group.

9.4. New best known packing-coverings. Finally we mention briefly a third application of our new theory. Closely related to the lattice covering problem, is the *lattice packing-covering problem*. It asks to minimize the *packing-covering constant*

$$\gamma(L) = \frac{\mu(L)}{\lambda(L)}$$

of a d -dimensional lattice L , where $\lambda(L) = \frac{1}{2} \min\{\|\mathbf{v}\| : \mathbf{v} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}\}$ is the so called *packing radius*. For a detailed description of this problem as well as for its interpretation as a convex optimization problem, we refer the interested reader to [SV06]. So

far, optimizing over suitable T -secondary cones with respect to 4-dimensional subspaces T of \mathcal{S}^7 , we found a new best known 7-dimensional packing-covering lattice with $\gamma(L) = 1.499399\dots$. The former record holder was the lattice E_7^* with $\gamma(E_7^*) = \sqrt{7/3} = 1.527525\dots$. It remains to undertake a systematic search for further best known, maybe optimal packing-covering lattices.

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MATHIEU DUTOUR SIKIRIĆ, RUDJER BOSKOVIĆ INSTITUTE, BIJENICKA 54, 10000 ZAGREB, CROATIA

E-mail address: mdsikir@irb.hr

ACHILL SCHÜRMANN, MATHEMATICS DEPARTMENT, UNIVERSITY OF MAGDEBURG, 39106 MAGDEBURG, GERMANY

E-mail address: achill@math.uni-magdeburg.de

FRANK VALLENTIN, CENTRUM VOOR WISKUNDE EN INFORMATICA (CWI), KRUISLAAN 413, 1098 SJ AMSTERDAM, THE NETHERLANDS

E-mail address: f.vallentin@cwi.nl